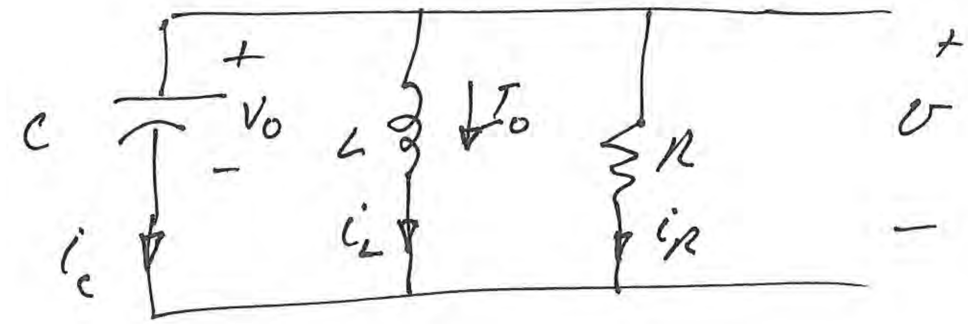
RLC CIRCUITS



$$\frac{v}{R} + \frac{1}{L}\int_0^t v(x)\,dx + I_0 + C\frac{dv}{dt} = 0$$

$$\frac{1}{R}\frac{dv}{dt} + \frac{v}{L} + C\frac{d^2v}{dt^2} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

HYPOTHESIZE A SOLUTION

$$v = Ae^{st}$$

DE BECOMES

$$As^2 e^{st} + \frac{As}{RC} e^{st} + \frac{Ae^{st}}{LC} = 0$$

$$Ae^{st}\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right) = 0$$

HOW TO GUARANTEE SOLUTION ?

$$A = 0 \; ?$$

$$e^{st} = 0 \; ?$$

$$s^2 + \frac{s}{RC} + \frac{1}{LC} = 0 \; ?$$

"CHARACTERISTIC EQUATION"

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

HOW MANY SOLUTIONS ?

ANY LINEAR COMBINATION OF SOLUTIONS

IS ALSO A SOLUTION:

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

ARCHITECTURE DETERMINES s_1 & s_2

INITIAL CONDITIONS DETERMINE A_1 & A_2

STANDARD FORM

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC}$$ "NEPER" FREQ, PARALLEL RLC

$$\omega_0 = \frac{1}{\sqrt{LC}}$$ RESONANT FREQ, PARALLEL RLC

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SCOTTISH MATHEMATICIAN

(INVENTER OF LOGARITHM)

✓ ASSESSMENT PROBLEM

Objective 1—Be able to determine the natural response and the step response of parallel *RLC* circuits

8.1 The resistance and inductance of the circuit in Fig. 8.5 are 100 Ω and 20 mH, respectively.

a) Find the value of C that makes the voltage response critically damped.

b) If C is adjusted to give a neper frequency of 5 krad/s, find the value of C and the roots of the characteristic equation.

c) If C is adjusted to give a resonant frequency of 20 krad/s, find the value of C and the roots of the characteristic equation.

Answer: (a) 500 nF;

(b) $C = 1\ \mu$F,
$s_1 = -5000 + j5000$ rad/s,
$s_2 = -5000 - j5000$ rad/s;

(c) $C = 125$ nF,
$s_1 = -5359$ rad/s,
$s_2 = -74{,}641$ rad/s.

NOTE: Also try Chapter Problem 8.4.

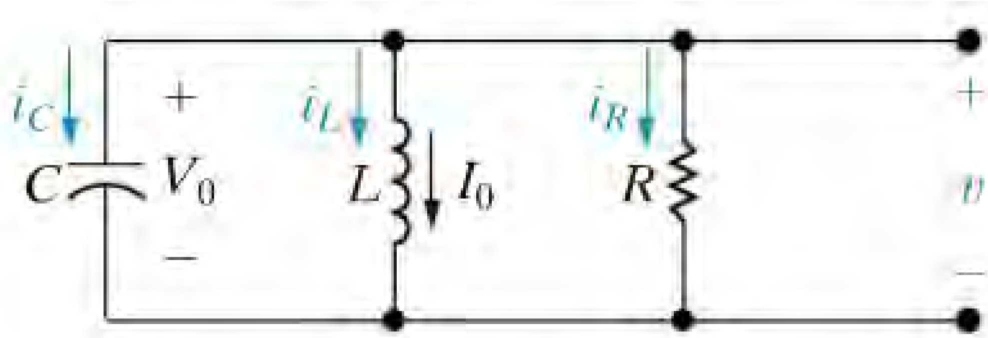


Figure 8.5 ▲ A circuit used to illustrate the natural response of a parallel *RLC* circuit.

$$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

ROOTS OF CHARACTERISTIC EQ

$\alpha = \frac{1}{2RC}$ "NEPER" FREQUENCY

$\omega_0 = \frac{1}{\sqrt{LC}}$ RESONANT FREQUENCY (UNITS RADIANS/TIME)

3 POSSIBLE CONDITIONS

$\omega_0^2 < \alpha^2$: ROOTS REAL & DISTINCT

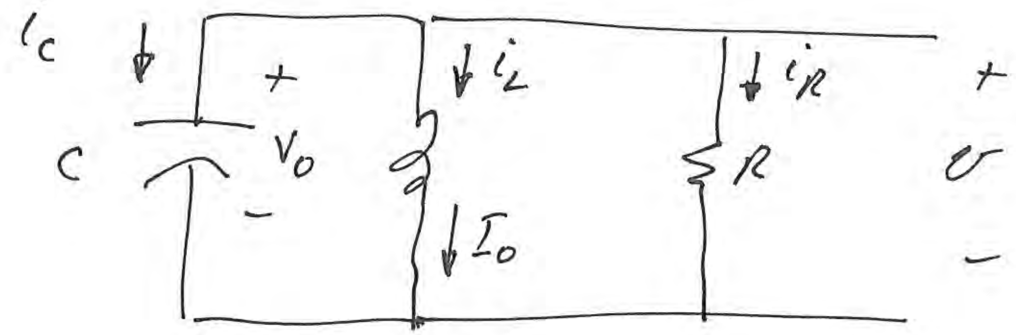
RESPONSE "OVERDAMPED"

$\omega_0^2 > \alpha^2$: ROOTS COMPLEX & CONJUGATES OF ONE ANOTHER

RESPONSE "UNDERDAMPED"

$\omega_0^2 = \alpha^2$: ROOTS REAL & IDENTICAL

RESPONSE "CRITICALLY DAMPED"



SOLUTION: $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

TWO UNKNOWNS $\Rightarrow$ NEED TWO EQUATIONS

EQ #1 $\quad v(0^+) = A_1 + A_2$ (KNOW THIS TO BE V_0)
WHY?

$$\frac{d}{dt} v(t) = A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}$$

$$\left.\frac{d}{dt} v(t)\right|_{t=0^+} = A_1 s_1 + A_2 s_2$$

APPLY KCL TO UPPER NODE:

$$i_c(0^+) + I_0 + \frac{V_0}{R} = 0$$

KNOWN

$$i_c(0^+) = -\frac{V_0}{R} - I_0$$

$$= C \left.\frac{dv(t)}{dt}\right|_{t=0^+}$$

EQ #2

$$C(A_1 S_1 + A_2 S_2) = -\frac{V_0}{R} - I_0$$

TWO EQ'S IN TWO UNKNOWNS YIELD SOLUTIONS FOR A_1 & A_2

$\therefore$ $v(t)$ IS SPECIFIED

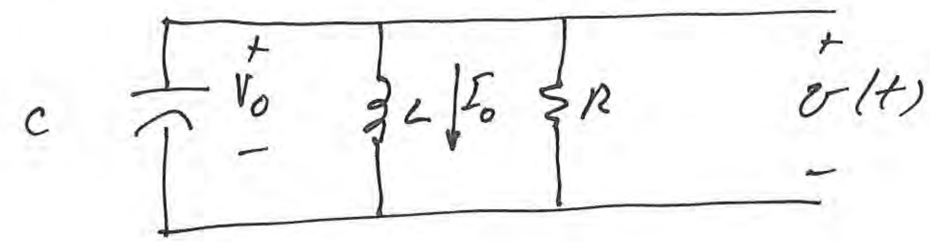
$\Longrightarrow$ BRANCH CURRENTS

$$i_R(t) = v(t)/R$$

$$i_L(t) = \frac{1}{L}\int_0^t v(x)\,dx + I_0$$

$$i_c(t) = C\,\frac{dv(t)}{dt}$$

EXAMPLE 8.2 PG 272



$C = 0.2\,\mu F$, $L = 50\,mH$, $R = 200\,\Omega$

$v(0^+) = 12\,V \;(= V_0)$

$i_L(0^+) = 30\,mA \;(= I_0)$

KCL: $i_c(0^+) + i_L(0^+) + \dfrac{v(0^+)}{R} = 0$

$$i_c(0^+) = -i_L(0^+) - \frac{v(0^+)}{R}$$

$$= -30\,mA - \frac{12\,V}{200\,\Omega} = -90\,mA$$

GENERAL SOL'N: $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

INITIAL COND'S: $A_1 + A_2 = V_0$ #1

$i_c(0^+) = C(A_1 s_1 + A_2 s_2)$ #2

$$s_1 = -12{,}500 + 7{,}500 = -5{,}000 \text{ RAD/S}$$

$$s_2 = -12{,}500 - 7{,}500 = -20{,}000 \text{ RAD/S}$$

(OVERDAMPED)

SUB EQ #1 INTO EQ #2

$$A_1 = \frac{i_c(0^+)/C - V_0 s_2}{s_1 - s_2} = -14 \text{ V}$$

EQ #1

$$A_2 = V_0 - A_1 = 12\text{ V} + 14\text{ V}$$

$$A_2 = 26 \text{ V}$$

$$v(t) = -14e^{-5{,}000t} + 26e^{-20{,}000t} \text{ V}; \ t \geq 0$$

$$i_R(t) = v(t)/R$$

$$i_L(t) = \frac{1}{L}\int_0^t v(x)\,dx + I_0$$

$$= \frac{1}{50\text{mH}}\left\{\frac{14}{5{,}000}e^{-5{,}000x}\Big|_0^t - \frac{26}{20{,}000}e^{-20{,}000x}\Big|_0^t\right\} + I_0$$

$$i_L(t) = 56e^{-5,000t} - 26e^{-20,000t} \text{ mA} \; ; \; t \geq 0$$

$$i_c(t) = C \frac{dv(t)}{dt}$$

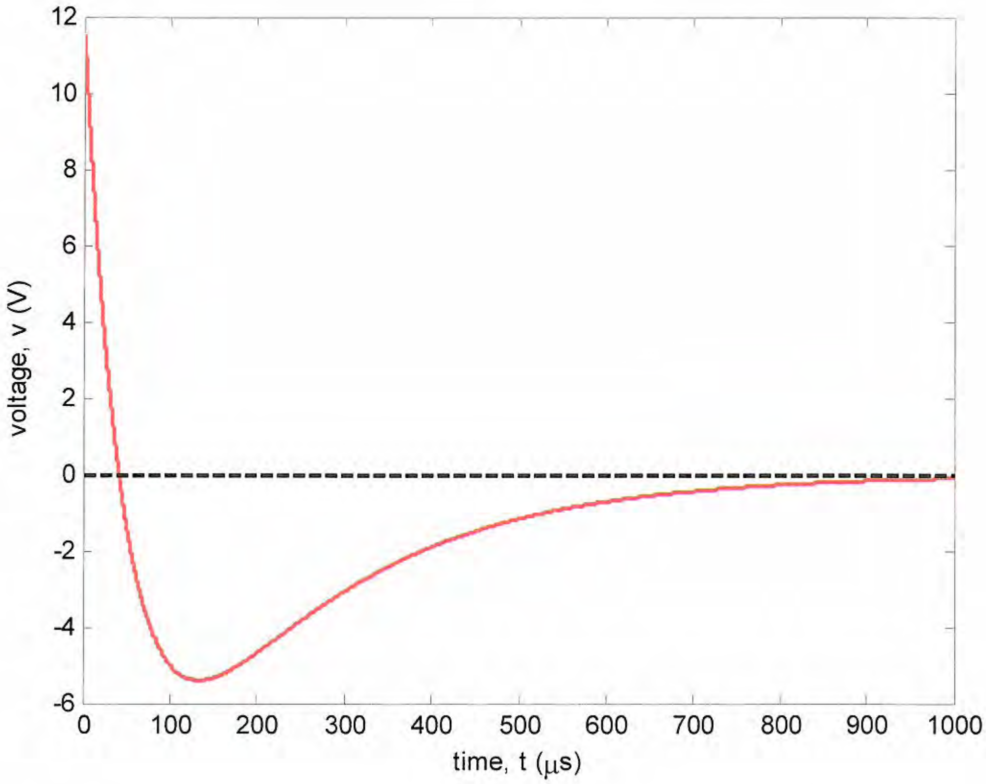
$$= 0.2\mu F \left\{ (14)(5,000)e^{-5,000t} - (26)(20,000)e^{-20,000t} \right\}$$

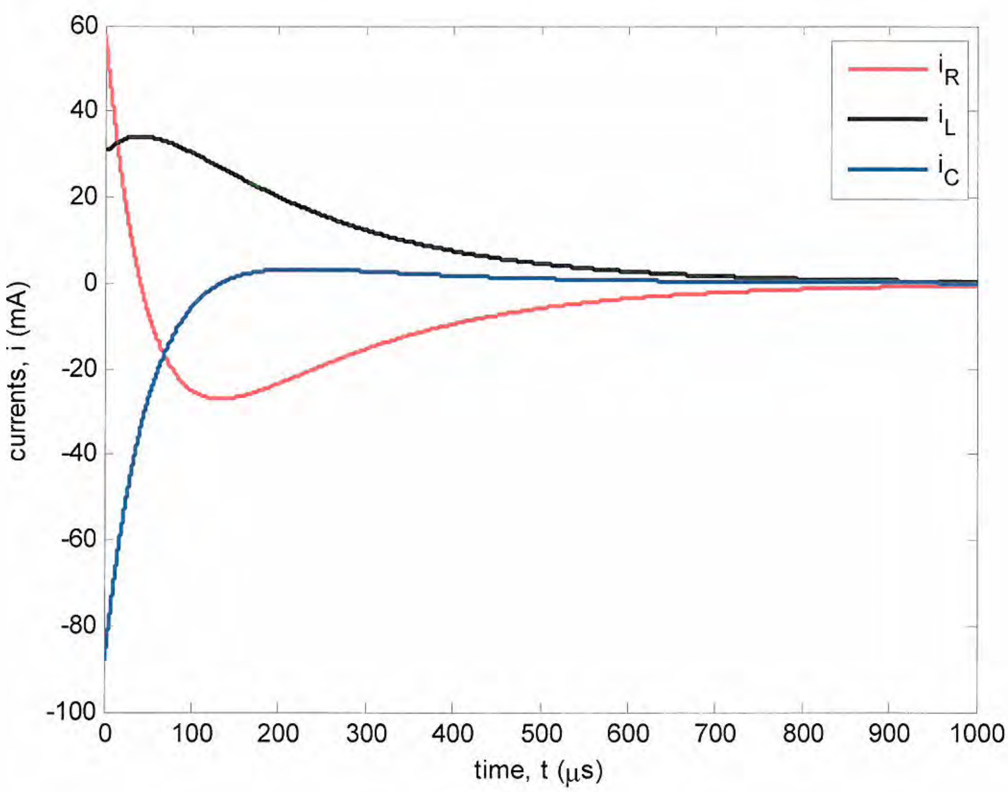
$$i_c(t) = 14e^{-5,000t} - 104e^{-20,000t} \; ; \; t \geq 0$$

mA

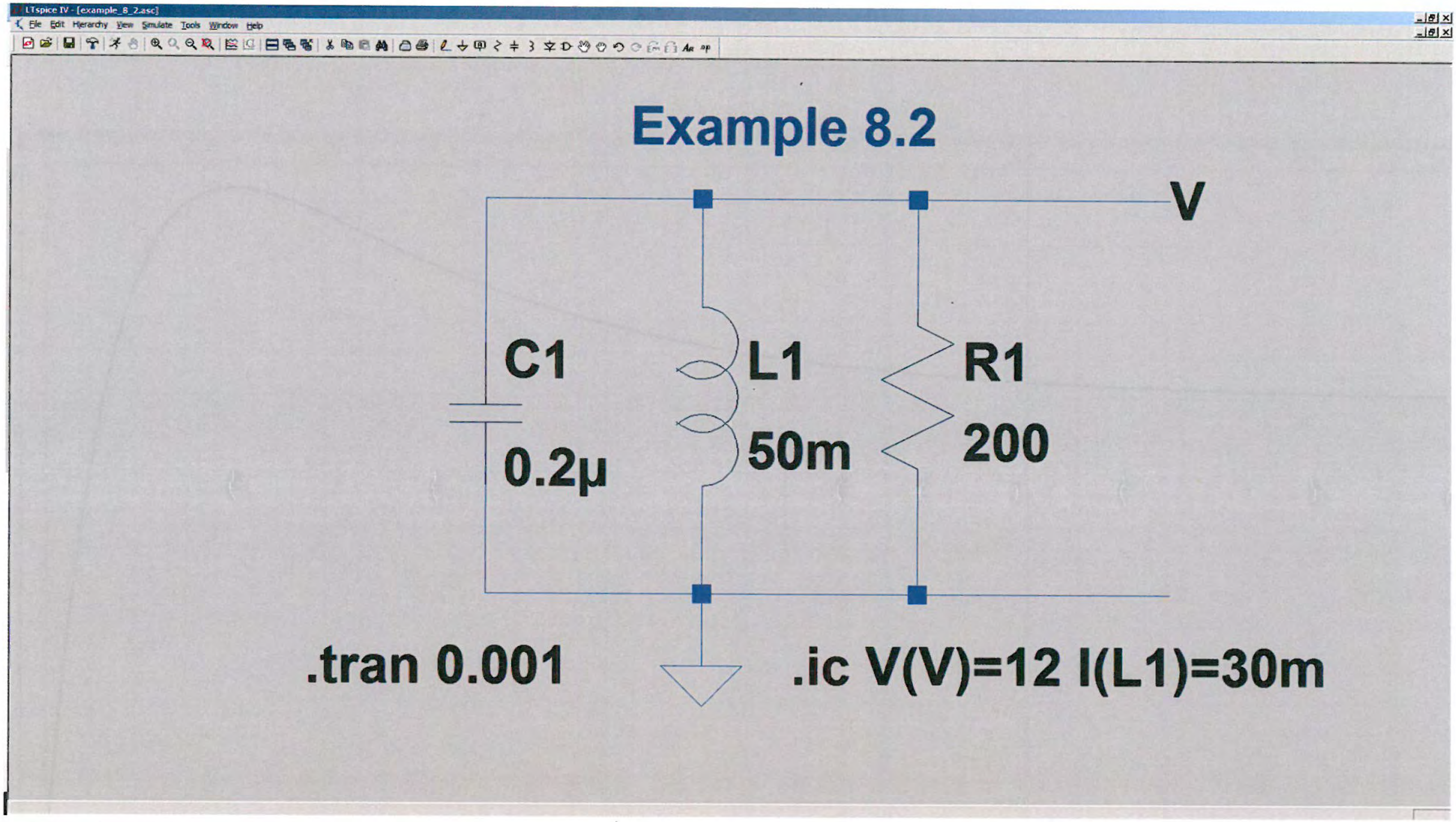
SHOULD BE

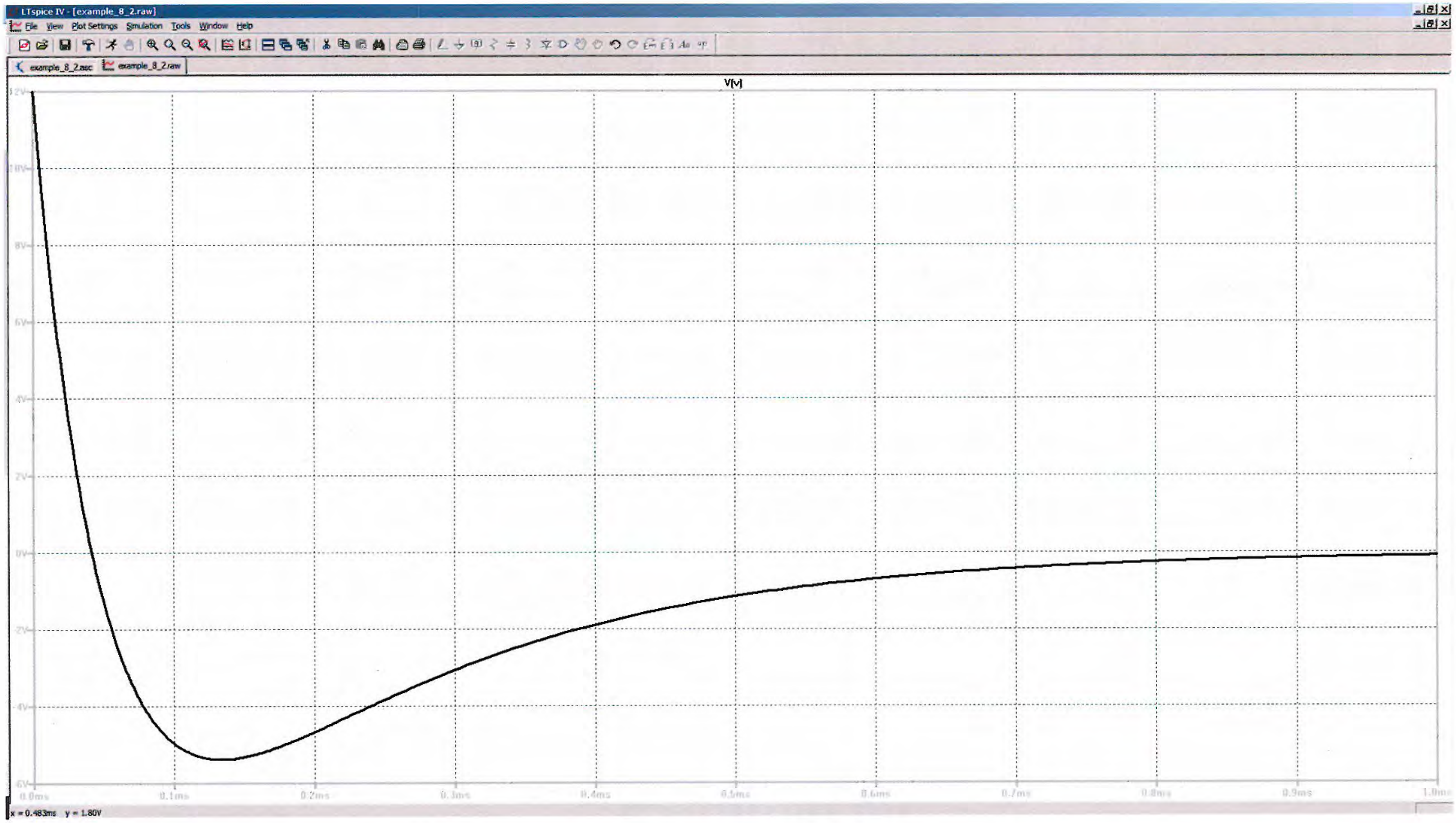
$t \geq 0^+$


voltage, v (V)
12
10
8
6
4
2
0
-2
-4
-6
0
100
200
300
400
500
600
700
800
900
1000
time, t (μs)

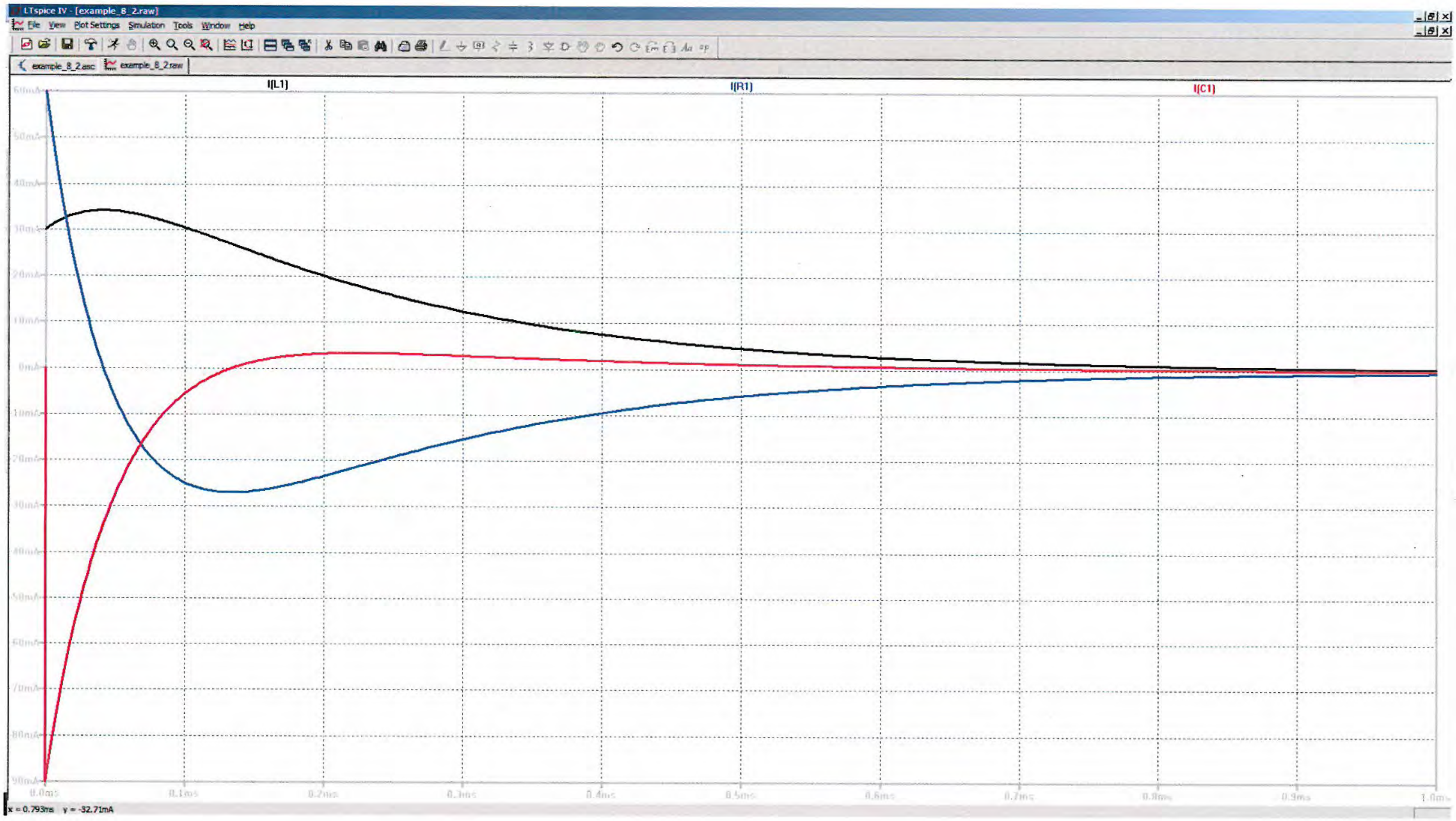

currents, i (mA)
60
40
20
0
-20
-40
-60
-80
-100
i_R
i_L
i_C
0
100
200
300
400
500
600
700
800
900
1000
time, t (μs)

```
% example_8_2
% 01/20/14 D D Duncan
%
N = 10000;
tmax = 1000;% max time in microsec
t = linspace(1,tmax,N)*1e-6;% time in sec
v = -14*exp(-5000*t) + 26*exp(-20000*t);
figure(1);plot(t*1e6,v,'r-',[0 tmax],[0 0],'k--');
xlabel('time, t (\mus)');ylabel('voltage, v (V)');
%
i_R = v/200;
i_L = 56e-3*exp(-5000*t) - 26e-3*exp(-20000*t);
i_C = 14e-3*exp(-5000*t) - 104e-3*exp(-20000*t);
figure(2);plot(t*1e6,i_R*1000,'r-',t*1e6,i_L*1000,'k',t*1e6,i_C*1000,'b-');
xlabel('time, t (\mus)');ylabel('currents, i (mA)');
legend('i_R','i_L','i_C')
```


LTspice IV - [example_8_2.asc]
File Edit Hierarchy View Simulate Tools Window Help
Example 8.2
V
C1
0.2µ
L1
50m
R1
200
.tran 0.001
.ic V(V)=12 I(L1)=30m



LTspice IV - [example_8_2.raw]
File View Plot Settings Simulation Tools Window Help
example_8_2.asc
example_8_2.raw
V(v)
12V
10V
8V
6V
4V
2V
0V
-2V
-4V
-6V
0.0ms
0.1ms
0.2ms
0.3ms
0.4ms
0.5ms
0.6ms
0.7ms
0.8ms
0.9ms
1.0ms
x = 0.483ms y = 1.80V



LTspice IV - [example_8_2.raw]
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example_8_2.asc
example_8_2.raw
I(L1)
I(R1)
I(C1)
x = 0.793ms y = -32.71mA

UNDERDAMPED CONDITION

$$s_i = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

UNDERDAMPING : $\omega_0^2 > \alpha^2$

$$s_i = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$$

$$= -\alpha \pm j\omega_d$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$ DAMPED RADIAN FREQUENCY

GENERAL SOLUTION:

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$= A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t}$$

$$\cancel{= (A_1 + A_2) e^{-\alpha t}}$$

RECALL EULER IDENTITY

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\Downarrow$$

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos\theta$$

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$$

$$v(t) = A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}$$

$$= A_1 e^{-\alpha t}(\cos\omega_d t + j\sin\omega_d t)$$

$$+ A_2 e^{-\alpha t}(\cos\omega_d t - j\sin\omega_d t)$$

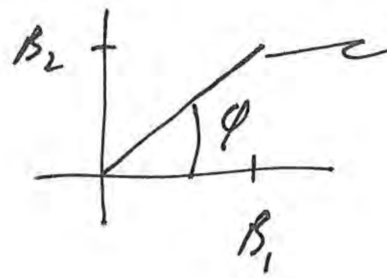
$$= e^{-\alpha t}\Big[(A_1 + A_2)\cos\omega_d t$$

$$+ j(A_1 - A_2)\sin\omega_d t\Big]$$

$$v(t) = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t$$

CAN BE WRITTEN MORE COMPACTLY:

$$v(t) = \sqrt{B_1^2 + B_2^2}\left[\frac{B_1}{\sqrt{B_1^2+B_2^2}}\cos\omega_d t + \frac{B_2}{\sqrt{B_1^2+B_2^2}}\sin\omega_d t\right]e^{-\alpha t}$$



$$\sqrt{B_1^2+B_2^2}$$

$$\varphi = \tan^{-1}\frac{B_2}{B_1}$$

$$\cos\varphi = \frac{B_1}{\sqrt{B_1^2+B_2^2}}$$

$$\sin\varphi = \frac{B_2}{\sqrt{B_1^2+B_2^2}}$$

$$v(t) = e^{-\alpha t}\sqrt{B_1^2+B_2^2}\,\underbrace{\left(\cos\varphi\cos\omega_d t + \sin\varphi\sin\omega_d t\right)}_{\cos(\omega_d t - \varphi)}$$

$$v(t) = \sqrt{B_1^2+B_2^2}\,e^{-\alpha t}\cos(\omega_d t - \varphi)$$

NOTE: B_i ARE REAL BECAUSE

$v(t)$ MUST BE REAL

$(\Rightarrow A_1 = A_2^*)$

INITIAL CONDITIONS DETERMINE B_i

$$v(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$$

$$\boxed{v(0^+) = B_1 = V_0}$$

$$i_c(0^+) = C \frac{dv}{dt}\bigg|_{t=0} = C \frac{d}{dt}\left(B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t \right)\bigg|_{t=0}$$

$$i_c(0^+) = C\left(-\alpha B_1 e^{-\alpha t} \cos \omega_d t - \omega_d B_1 e^{-\alpha t} \sin \omega_d t - \alpha B_2 e^{-\alpha t} \sin \omega_d t + \omega_d B_2 e^{-\alpha t} \cos \omega_d t \right)\bigg|_{t=0}$$

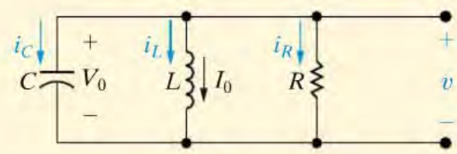
$$\boxed{i_c(0^+) = (-\alpha B_1 + \omega_d B_2)\, C}$$

✓ASSESSMENT PROBLEMS

Objective 1—Be able to determine the natural response and the step response of parallel *RLC* circuits

8.2 Use the integral relationship between i_L and v to find the expression for i_L in Fig. 8.6.

Answer: $i_L(t) = (56e^{-5000t} - 26e^{-20{,}000t})$ mA, $t \geq 0$.



8.3 The element values in the circuit shown are $R = 2\text{ k}\Omega$, $L = 250$ mH, and $C = 10$ nF. The initial current I_0 in the inductor is -4 A, and the initial voltage on the capacitor is 0 V. The output signal is the voltage v. Find (a) $i_R(0^+)$; (b) $i_C(0^+)$; (c) $dv(0^+)/dt$; (d) A_1; (e) A_2; and (f) $v(t)$ when $t \geq 0$.

Answer: (a) 0;
(b) 4 A;
(c) 4×10^8 V/s;
(d) 13,333 V;
(e) $-13{,}333$ V;
(f) $13{,}333(e^{-10{,}000t} - e^{-40{,}000t})$ V.

NOTE: Also try Chapter Problems 8.5 and 8.13.

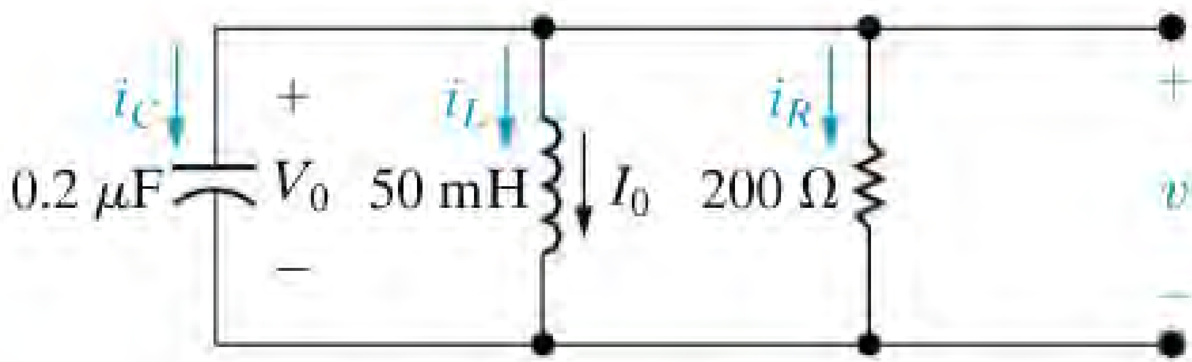


Figure 8.6 ▲ The circuit for Example 8.2.

OVERDAMPED

$$v(t) = A_1 e^{-s_1 t} + A_2 e^{-s_2 t}$$

$$v(0) = A_1 + A_2 = v_C(0^+)$$

$$i_C(0^+) = C \frac{dv}{dt}\Big|_{t=0} = C(A_1 s_1 + A_2 s_2)$$

$$= -i_L(0^+) - \frac{v(0^+)}{R}$$

UNDERDAMPED

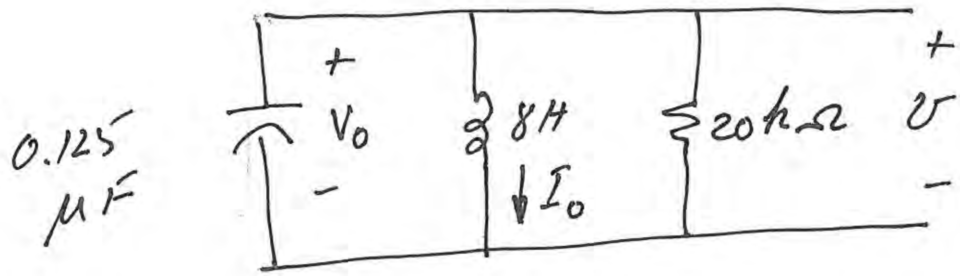
$$v(t) = (B_1 \cos \omega_d t + B_2 \sin \omega_d t) e^{-\alpha t}$$

$$v(0) = B_1 = v_C(0^+)$$

$$i_C(0^+) = (-\alpha B_1 + \omega_d B_2) C = -i_L(0^+) - \frac{v(0^+)}{R}$$

CHEATSHEET
FOR
ASSESSMENT 8.2

EXAMPLE 8.4



$V_0 = 0$

$I_0 = -12.25mA$

$\alpha = \frac{1}{2RC} = 200 \text{ RAD/S}$

$\omega_0 = \frac{1}{\sqrt{LC}} = 1{,}000 \text{ RAD/S}$

$\omega_0 > \alpha \Rightarrow$ UNDERDAMPED

$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 979.8 \text{ RAD/S}$

$v(t) = B_1 e^{-\alpha t} \cos\omega_d t + B_2 e^{-\alpha t} \sin\omega_d t$

INITIAL COND'S.

$$v(0) = V_0 = B_1 = 0$$

$$i_c(0^+) = C\omega_d B_2$$

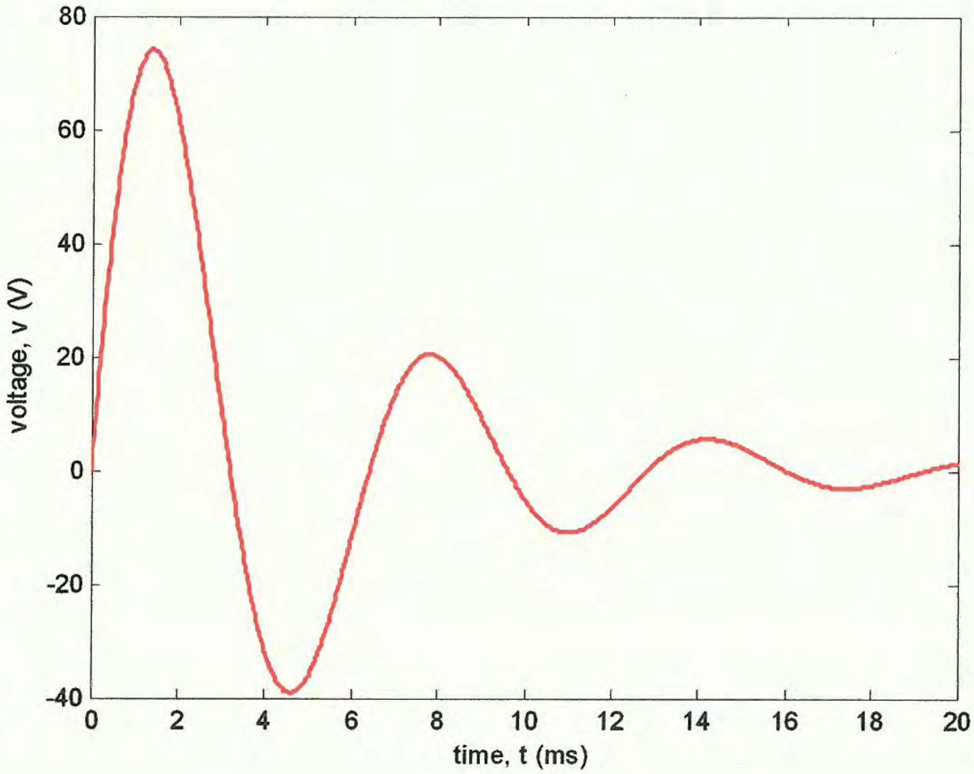
KCL: $\underset{\substack{\uparrow \\ I_0}}{i_L(0^+)} + i_c(0^+) + \underset{\substack{\uparrow \\ \frac{V_0}{R} = 0}}{i_R(0^+)} = 0$

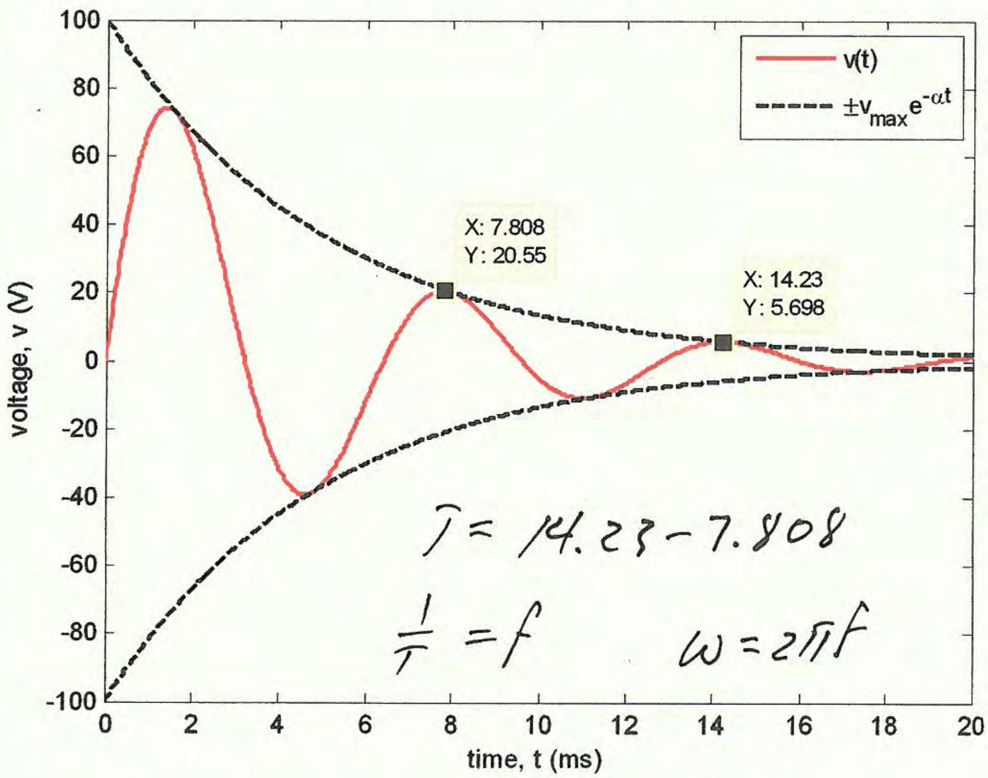
$$i_c(0^+) = -I_0 = C\omega_d B_2$$

$$B_2 = \frac{-I_0}{C\omega_d} = \frac{12.25\,mA}{(0.125\,\mu F)(979.8\,RAD/S)}$$

$$B_2 \simeq 100\,V$$

$$v(t) = 100e^{-200t} \sin(979.8t)\,V \;;\; t \geq 0$$



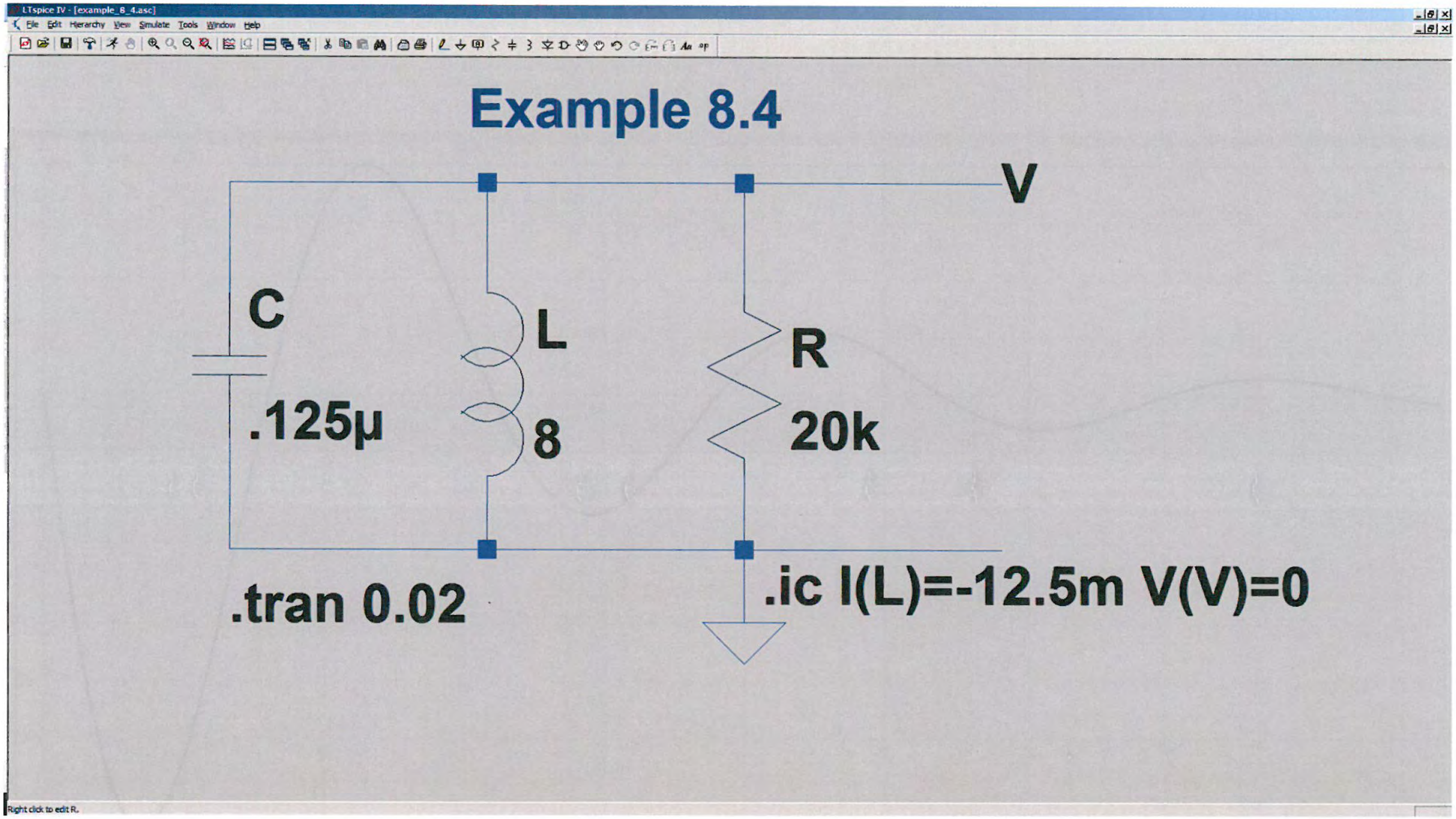


$T = 14.23 - 7.808$

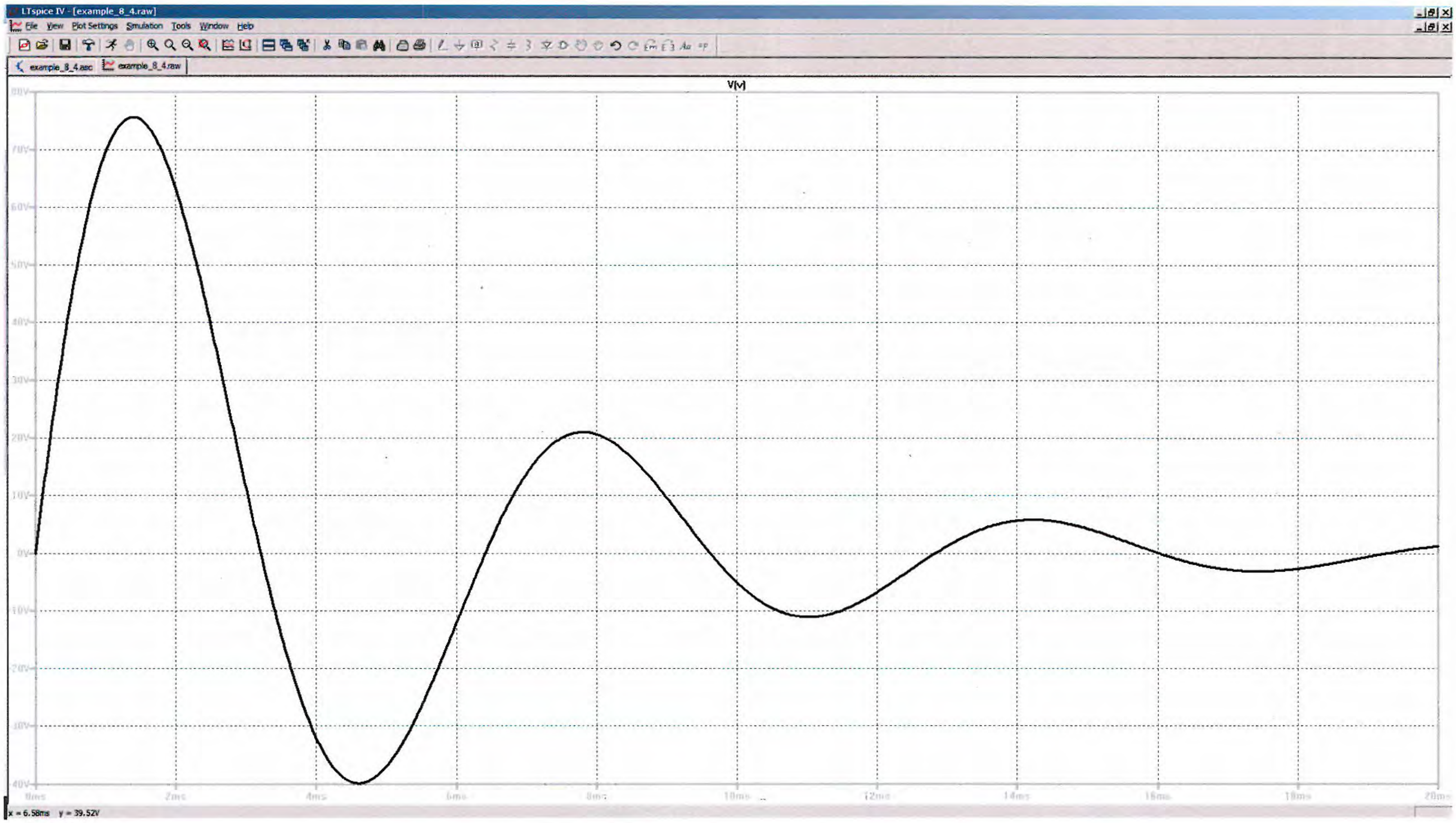
$\frac{1}{T} = f \qquad \omega = 2\pi f$

$$\omega_d = \frac{2\pi}{(14.23 - 7.808)\,\text{ms}}$$

```
% example_8_4
% 01_20_14 D D Duncan
%
alpha = 200;
omega_d = 100*sqrt(96);
tmax = 20e-3;% max time in sec
vmax = 100;
N = 1000;
t = linspace(0,tmax,N);% time in sec
v = vmax*exp(-alpha*t).*sin(omega_d*t);% voltage in volts
figure(1);plot(t*1000,v,'r-');
xlabel('time, t (ms)');ylabel('voltage, v (V)');
envelope = vmax*exp(-alpha*t);
figure(2);plot(t*1000,v,'r-',t*1000,envelope,'k--',t*1000,-envelope,'k--');
xlabel('time, t (ms)');ylabel('voltage, v (V)');
legend('v(t)','\pmv_{max}e^{-\alphat}');
```



LTspice IV - [example_8_4.asc]
File Edit Hierarchy View Simulate Tools Window Help
Example 8.4
V
C
.125µ
L
8
R
20k
.tran 0.02
.ic I(L)=-12.5m V(V)=0
Right click to edit R.



LTspice IV - [example_8_4.raw]
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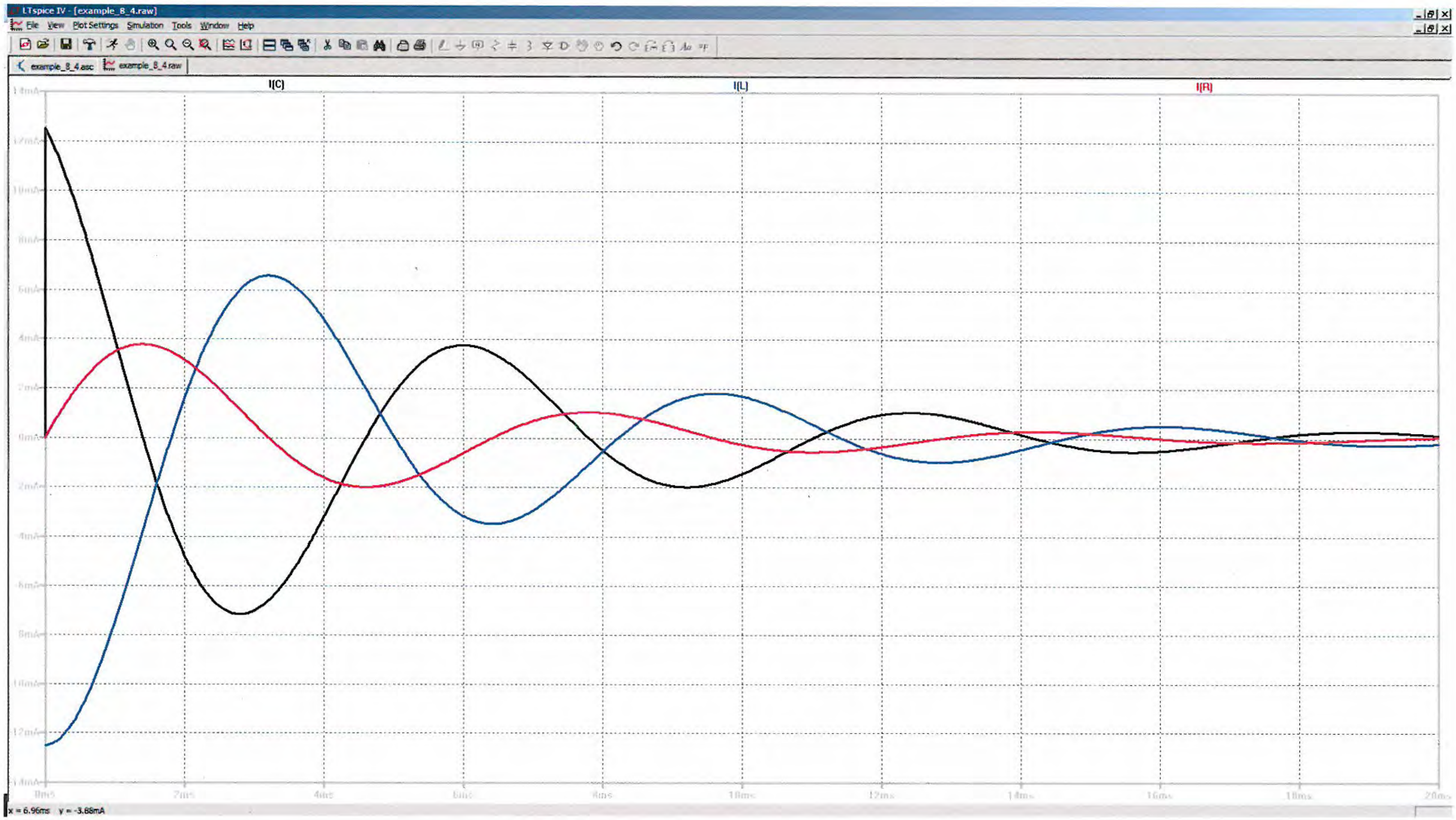
example_8_4.asc
example_8_4.raw
V(v)
80V
70V
60V
50V
40V
30V
20V
10V
0V
-10V
-20V
-30V
-40V
0ms
2ms
4ms
6ms
8ms
10ms
12ms
14ms
16ms
18ms
20ms
x = 6.58ms y = 39.52V


LTspice IV - [example_8_4.raw]
File View Plot Settings Simulation Tools Window Help
example_8_4.asc
example_8_4.raw
I(C)
I(L)
I(R)
x = 6.96ms y = -3.88mA

CRITICAL DAMPING

RECALL CRITICAL DAMPING $\Rightarrow \alpha = \omega_0$

$$\Rightarrow s_1 = s_2 = -\alpha = -\frac{1}{2RC}$$

$\Rightarrow$ GENERAL SOL'N. $v(t) = (A_1 + A_2)e^{-\alpha t}$

$$v(t) = Ae^{-\alpha t} \;?$$

ONLY 1 ARBITRARY CONSTANT

$\Rightarrow$ CANNOT SATISFY NECESSARY (TWO) BOUNDARY, i.e., INITIAL CONDITIONS

$\Rightarrow$ MUST MODIFY HYPOTHESIZED SOLUTION FORM

$$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

ORIGINAL D.E.

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

$\alpha = \frac{1}{2RC}$, CRITICAL DAMPING $\Rightarrow \left(\frac{1}{2RC}\right) = \frac{1}{\sqrt{LC}}$

$= \alpha$

$$\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \alpha^2 v = 0$$

SHOULD BE ABLE TO SHOW THAT

$$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

IS A SOLUTION

INITIAL CONDITIONS

$$\boxed{v(0) = D_2 = V_0}$$

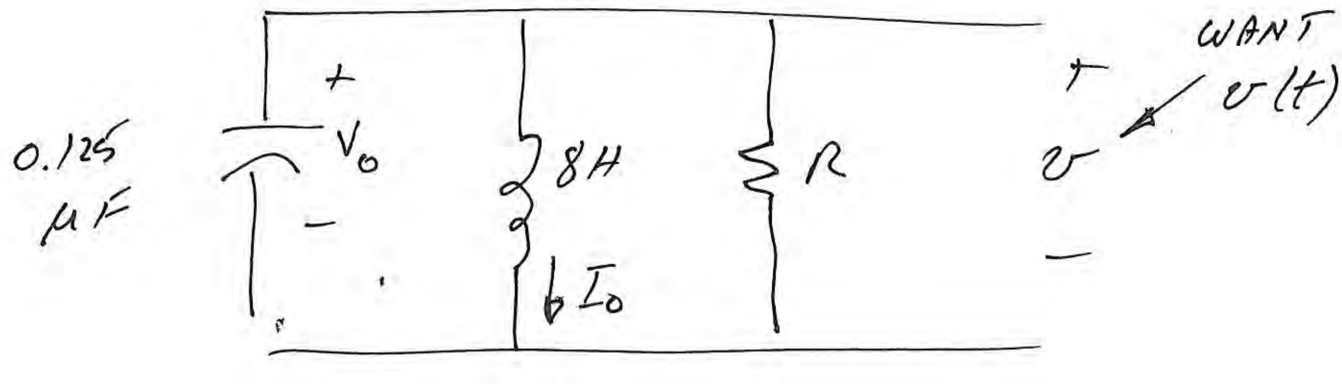
$$i_c(0^+) = C \frac{dv}{dt}\Big|_{t=0^+}$$

$$= C \frac{d}{dt}\left[(D_1 t + D_2)e^{-\alpha t}\right]\Big|_{t=0}$$

$$= C\left[D_1 e^{-\alpha t} - \alpha(D_1 t + D_2)e^{-\alpha t}\right]\Big|_{t=0}$$

$$\boxed{i_c(0^+) = C(D_1 - \alpha D_2)}$$

EXAMPLE 8.5



FROM EX 8.4, $\omega_0 = \frac{1}{\sqrt{LC}} = 10^3$ RAD/S

$$\alpha = \frac{1}{2RC}$$

FOR CRITICAL DAMPING, $\alpha = \omega_0 = 10^3$ RAD/S

$$\Rightarrow R = \frac{1}{2\alpha C} = 4k\Omega$$

FROM EX 8.4 $v_0(0^+) = 0$, $I_0 = -12.25\, mA$

GENERAL FORM:

$$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

$$v(0) = D_2 = v_0(0^+) \Rightarrow D_2 = 0$$

$$\Rightarrow v(t) = D_1 t e^{-\alpha t}$$

OTHER I.C. IS

$$i_c(0^+) = C(D_1 - \alpha D_2) = C D_1$$

KCL: $$i_c(0^+) + I_0 + \cancel{\frac{V_0}{R}}^{\,0} = 0$$

$$i_c(0^+) = -I_0$$

$$-I_0 = C D_1 \longrightarrow D_1 = \cancel{0.1225\,mA}\ \frac{12.25\,mA}{0.125\,\mu F}$$

$$D_1 = 98{,}000$$ WHAT ARE UNITS?

$$D_1 = \frac{-I_0}{C}$$

$$[D_1] = \frac{A}{F} = \frac{A}{C/V} = \frac{C/S}{C/V} = \frac{V}{S}$$

$$[D_1] = \frac{V}{S}$$

$$v(t) = 98{,}000\, t e^{-1000t}\ V;\ t \geq 0$$

t ~~IS~~ IN S

$$v(t) = 98\, t e^{-t}\ V;\ t \geq 0$$

t IN mS

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

WHAT HAPPENS WHEN $S_1 = S_2$?

LET $\dfrac{S_1 + S_2}{2} = S$, $S_1 - S_2 = \epsilon$ (SMALL)

$$S_1 = S + \epsilon/2 \quad , \quad S_2 = S - \epsilon/2$$

SUBSTITUTE INTO SOLUTION FORM

$$v(t) = A_1 e^{(s + \epsilon/2)t} + A_2 e^{(s - \epsilon/2)t}$$

$$= A_1 e^{st} e^{\epsilon t/2} + A_2 e^{st} e^{-\epsilon t/2}$$

USE APPROXIMATION $e^{\pm \epsilon t/2} \approx 1 \pm \frac{\epsilon t}{2}$

$$v(t) = A_1 e^{st}(1 + \epsilon t/2) + A_2 e^{st}(1 - \epsilon t/2)$$

COLLECT TERMS

$$v(t) = \frac{(A_1 - A_2)\epsilon}{2} t e^{st} + (A_1 + A_2) e^{st}$$

$$v(t) = D_1 t e^{st} + D_2 e^{st}$$

CRITICAL DAMPING

$$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

$$v(0) = D_2 = V_0$$

$$i_c(0^+) = C \frac{dv(t)}{dt}\bigg|_{t=0^+} = C(D_1 - \alpha D_2)$$

KCL GIVES $\quad C(D_1 - \alpha D_2) = -\underset{\substack{\uparrow \\ i_L(0^+)}}{I_0} - \dfrac{v(0^+)}{R}.$

✓ ASSESSMENT PROBLEM

Objective 1—Be able to determine the natural and the step response of parallel *RLC* circuits

8.5 The resistor in the circuit in Assessment Problem 8.4 is adjusted for critical damping. The inductance and capacitance values are 0.4 H and 10 μF, respectively. The initial energy stored in the circuit is 25 mJ and is distributed equally between the inductor and capacitor. Find (a) R; (b) V_0; (c) I_0; (d) D_1 and D_2 in the solution for v; and (e) i_R, $t \geq 0^+$.

Answer: (a) 100 Ω;

(b) 50 V;

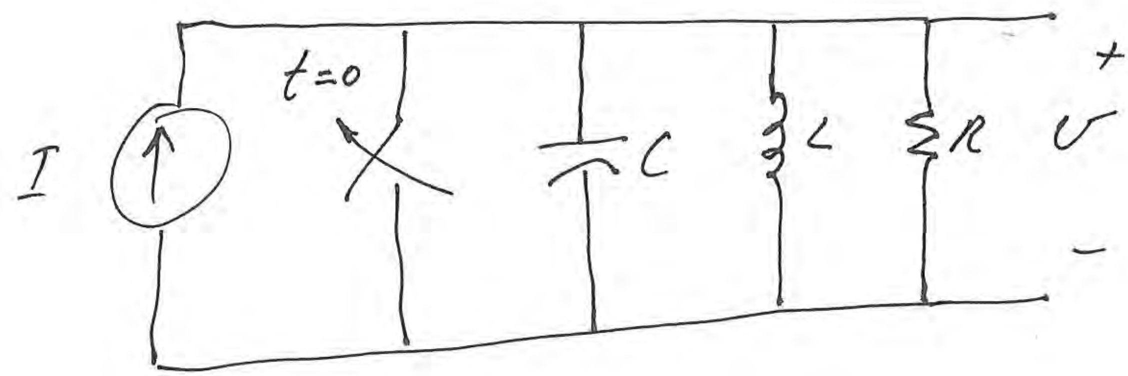
(c) 250 mA;

(d) −50,000 V/s, 50 V;

(e) $i_R(t) = (-500te^{-500t} + 0.50e^{-500t})$ A, $t \geq 0^+$.

NOTE: Also try Chapter Problems 8.7 and 8.12.

STEP RESPONSE PARALLEL RLC



KCL: $i_L + i_R + i_C = I$

$$i_L + \frac{v}{R} + C\frac{dv}{dt} = I$$

$$\vdots$$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}$$

SOLVE BY KNOWN METHODS

("DIRECT" APPROACH)

NOTE: SOL'N = FORCED RESP.

+ NATURAL RESP.

MISSING STEPS PREVIOUS PAGE

$$i_L + \frac{v}{R} + C\frac{dv}{dt} = I$$

$$v_L = L\frac{di_L}{dt} = v$$

$$\Rightarrow \frac{v}{R} = \frac{L}{R}\frac{di_L}{dt}$$

$$\Rightarrow L\frac{d^2 i_L}{dt^2} = \frac{dv}{dt}$$

$$\Rightarrow C\frac{dv}{dt} = CL\frac{d^2 i_L}{dt^2}$$

$$i_L + \frac{L}{R}\frac{di_L}{dt} + CL\frac{d^2 i_L}{dt^2} = I$$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}$$

"INDIRECT" APPROACH

$$i_L + \frac{v}{R} + C\frac{dv}{dt} = I$$

$$\frac{1}{L}\int_0^t v(\tau)d\tau + \frac{v}{R} + C\frac{dv}{dt} = I$$

$\frac{d}{dt}$ BOTH SIDES:

$$\frac{v}{L} + \frac{1}{R}\frac{dv}{dt} + C\frac{d^2v}{dt^2} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

SOLVE BY KNOWN METHODS:

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$v = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t$$

$$v = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

THESE ARE NATURAL RESPONSES...

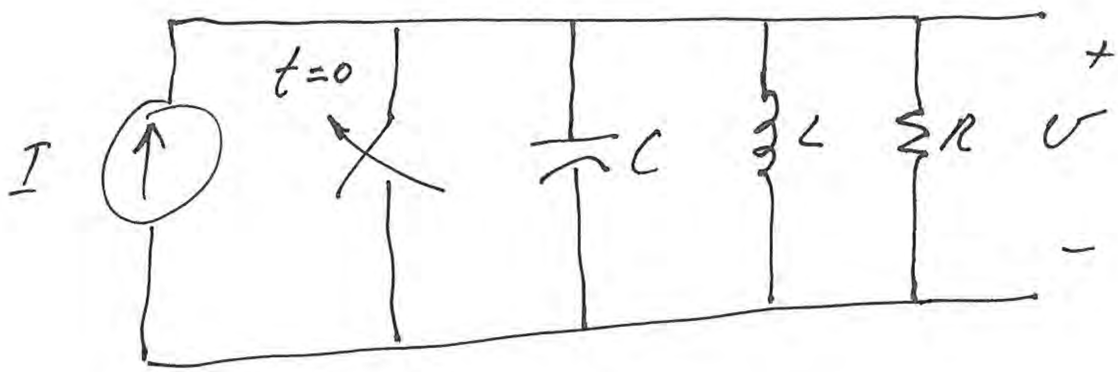
RECALL

$$i_L + \frac{v}{R} + C\frac{dv}{dt} = I$$

USE SOL'N. FOR v, E.G., OVERDAMPED ...

$$i_L = I + A_1' e^{s_1 t} + A_2' e^{s_2 t}$$

STEP RESPONSE PARALLEL RLC



KCL: $i_L + i_R + i_C = I$

$$i_L + \frac{v}{R} + C\frac{dv}{dt} = I$$

$$\vdots$$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}$$

SOLVE BY KNOWN METHODS

("DIRECT" APPROACH)

NOTE: SOL'N = FORCED RESP.

+ NATURAL RESP.

MISSING STEPS PREVIOUS PAGE

$$i_L + \frac{v}{R} + C\frac{dv}{dt} = I$$

$$v_L = L\frac{di_L}{dt} = v$$

$$\Rightarrow \frac{v}{R} = \frac{L}{R}\frac{di_L}{dt}$$

$$\Rightarrow L\frac{d^2 i_L}{dt^2} = \frac{dv}{dt}$$

$$\Rightarrow C\frac{dv}{dt} = CL\frac{d^2 i_L}{dt^2}$$

$$i_L + \frac{L}{R}\frac{di_L}{dt} + CL\frac{d^2 i_L}{dt^2} = I$$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}$$

"INDIRECT" APPROACH

$$i_L + \frac{v}{R} + C\frac{dv}{dt} = I$$

$$\frac{1}{L}\int_0^t v(\tau)d\tau + \frac{v}{R} + C\frac{dv}{dt} = I$$

$\frac{d}{dt}$ BOTH SIDES:

$$\frac{v}{L} + \frac{1}{R}\frac{dv}{dt} + C\frac{d^2v}{dt^2} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

SOLVE BY KNOWN METHODS:

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$v = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t$$

$$v = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

THESE ARE NATURAL RESPONSES...

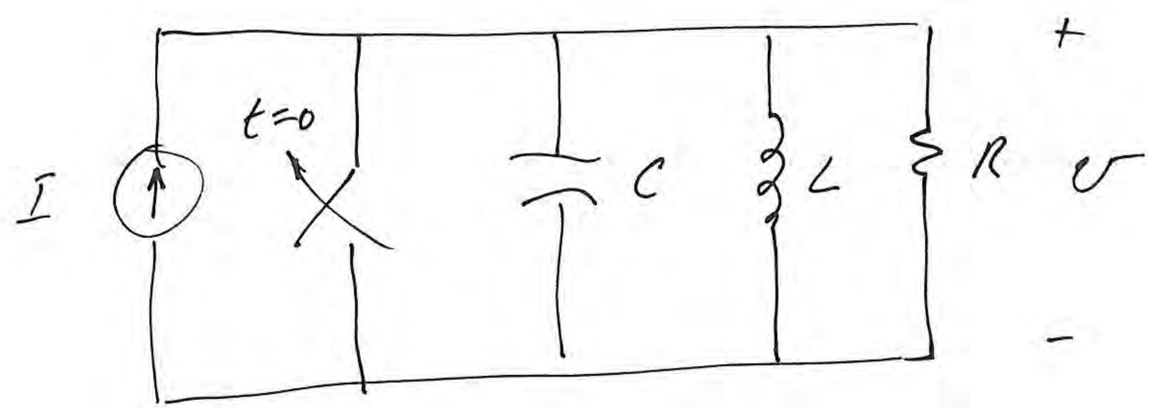
RECALL

$$i_L + \frac{v}{R} + C\frac{dv}{dt} = I$$

USE SOL'N. FOR v, E.G., OVERDAMPED...

$$i_L = I + A_1' e^{s_1 t} + A_2' e^{s_2 t}$$

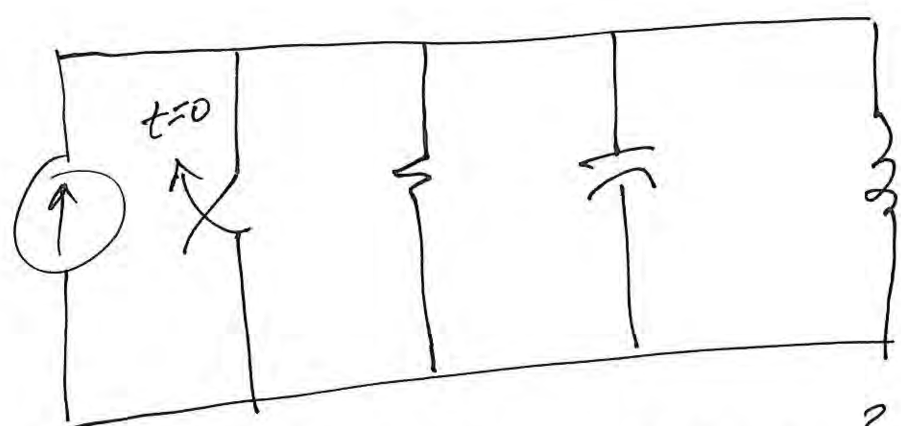
STEP RESPONSE // RLC CKT



NO ENERGY STORED PRIOR TO $t=0$

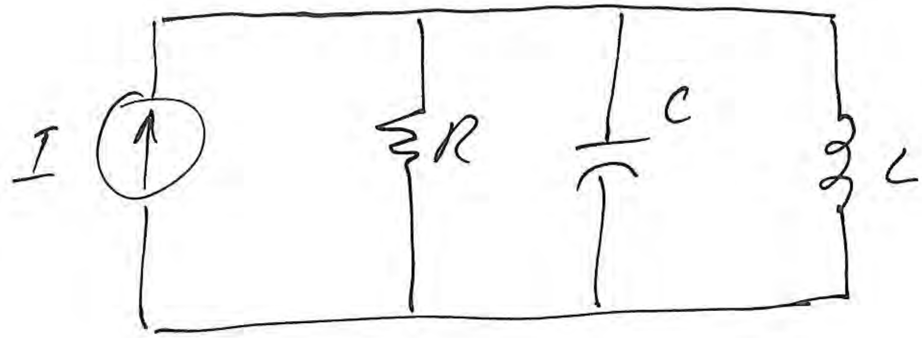
$\Longrightarrow \quad v(0) = 0$

$\quad\quad i_L(0) = 0$



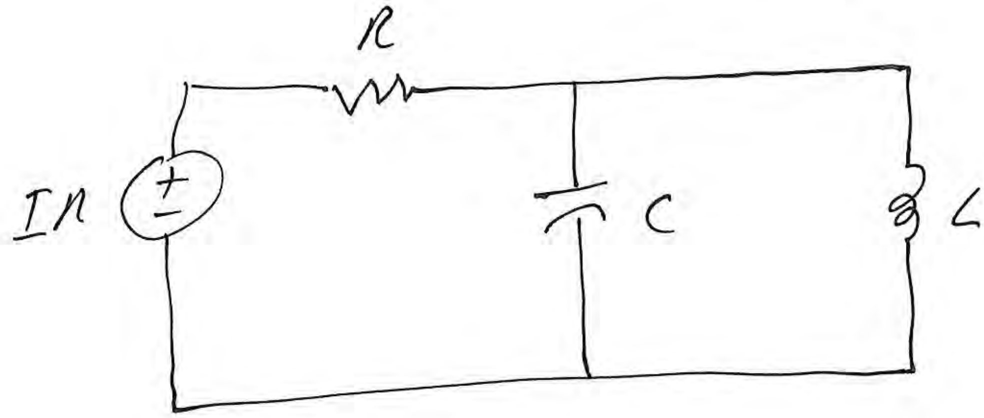
CKT EQUIVALENT ?

$t \geq 0$

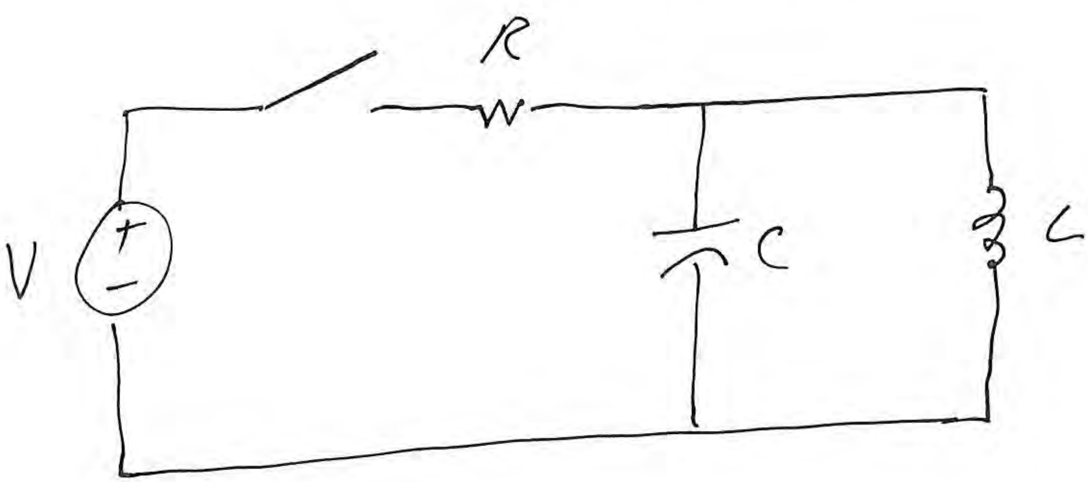


SAME AS

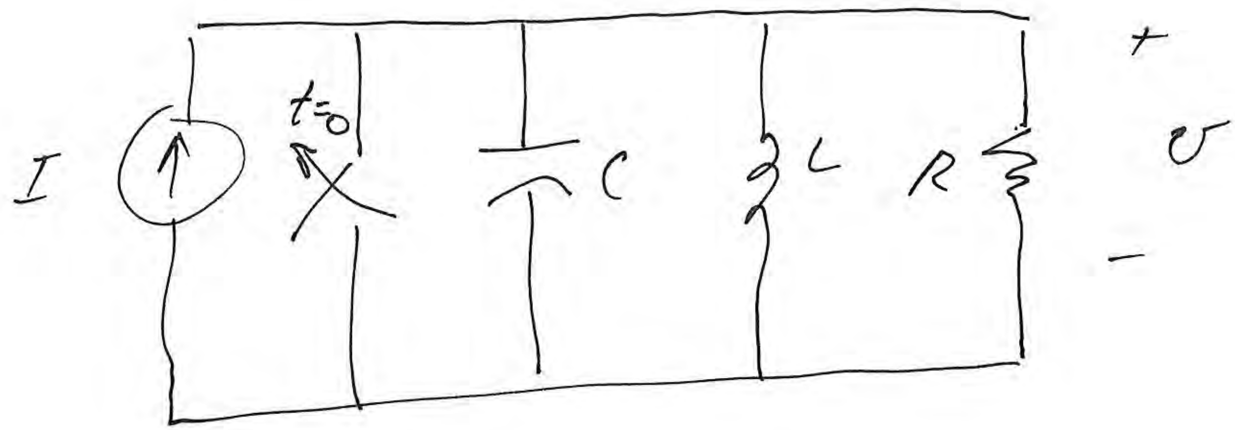
$t \geq 0$



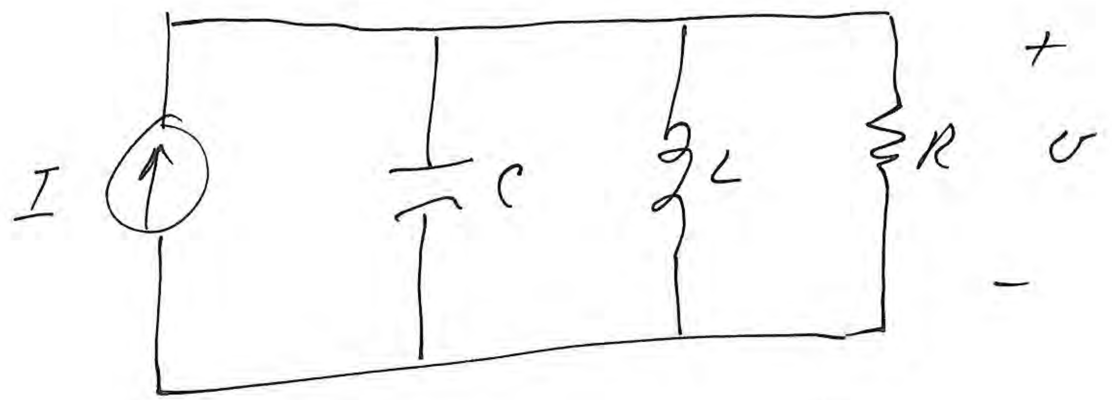
EQUIVALENT PROBLEM



WHAT IS SWITCH OPERATION
?



$t \geq 0$



KCL: $i_L + i_R + i_C = I$

$$i_L + \frac{v}{R} + C\frac{dv}{dt} = I$$

RECALL $v = L\dfrac{di_L}{dt} \Rightarrow \dfrac{dv}{dt} = L\dfrac{d^2 i_L}{dt^2}$

$$i_L + \frac{L}{R}\frac{di_L}{dt} + LC\frac{d^2 i_L}{dt^2} = I$$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}$$

NOTE INHOMOGENEOUS D.E.

AS $t \to \infty$ $\quad i_L(t) \longrightarrow i_L(\infty)$ (CONSTANT)

$\Rightarrow$ AS $t \to \infty$ $\quad \frac{di_L}{dt} \longrightarrow 0$

$\Rightarrow$ D.E. BECOMES

$$\frac{i_L(\infty)}{LC} = \frac{I}{LC}$$

$$\boxed{i_L(\infty) = I}$$

WHAT DOES THIS IMPLY?

COMPLETE SOLUTION OF D.E. =

HOMOGENEOUS SOLUTION

+

PARTICULAR SOLUTION

NATURAL RESPONSE

FORCED RESPONSE

WHAT IS THIS FORCED RESPONSE ?

$i_L(t) = I_f +$ NATURAL RESPONSE

OVER DAMPED

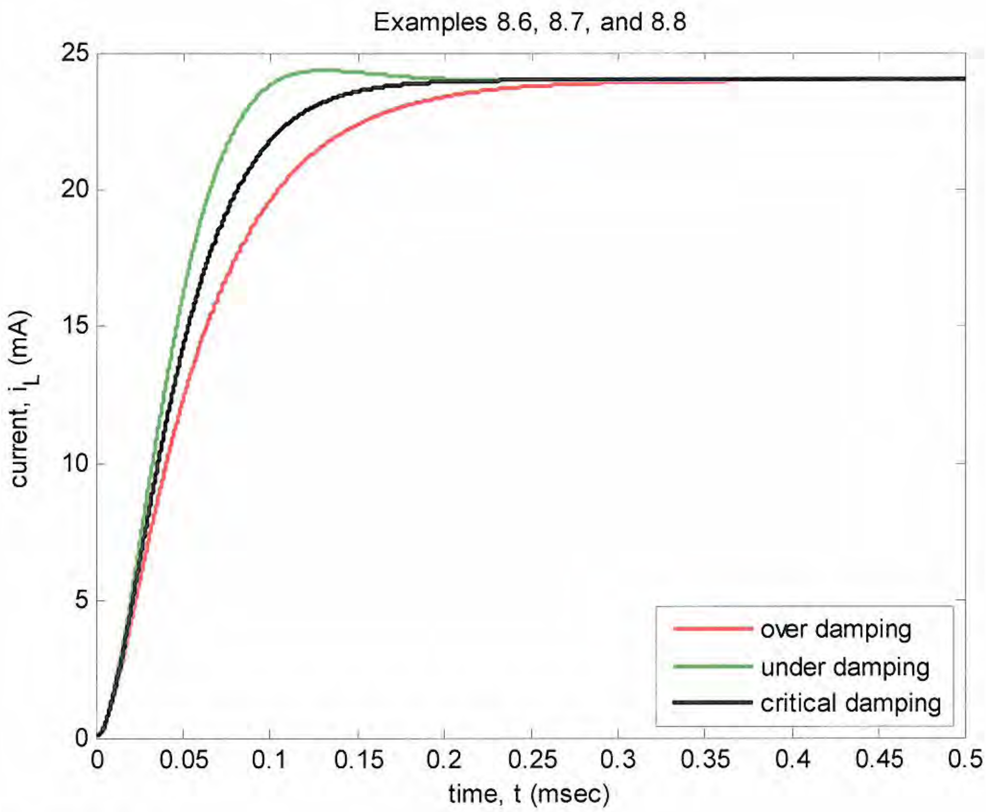
$$i_L = I_f + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

UNDER DAMPED

$$i_L = I_f + e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

CRITICALLY DAMPED

$$i_L = I_f + D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$



over damping

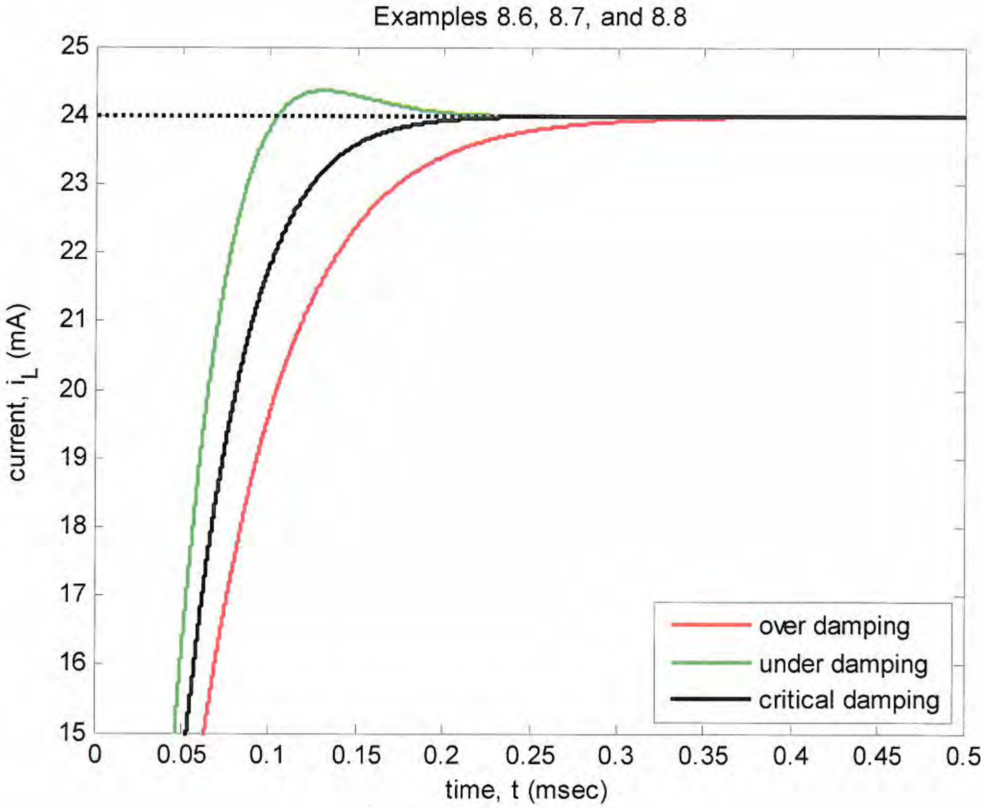
$$i_L(t) = 24 - 32e^{-20,000t} + 8e^{-80,000t} \quad \text{mA}$$

under damping

$$i_L(t) = 24 - e^{-32,000t}\left(24\cos 24,000t + 32\sin 24,000t\right) \quad \text{mA}$$

critical damping

$$i_L(t) = 24 - e^{-40,000t}\left(960,000t + 24\right) \quad \text{mA}$$



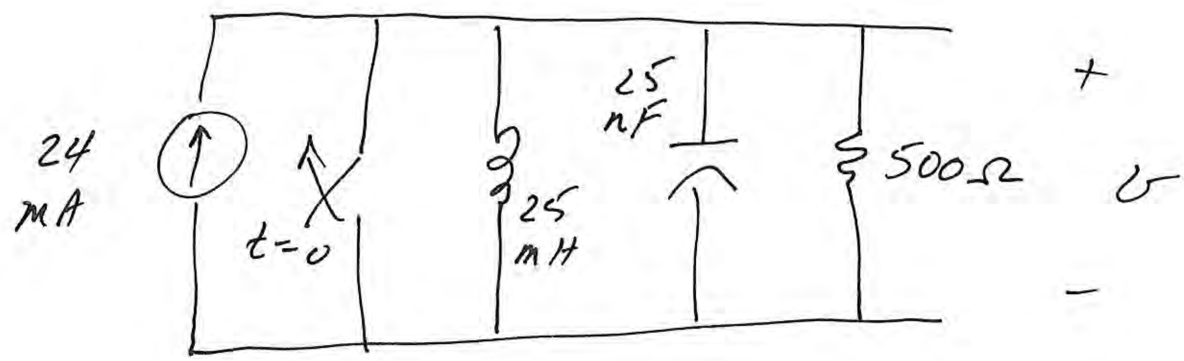
Examples 8.6, 8.7, and 8.8
current, i_L (mA)
25
24
23
22
21
20
19
18
17
16
15
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
time, t (msec)
over damping
under damping
critical damping

EXAMPLES 8.6 – 8.9 HAVE NO STORED ENERGY PRIOR TO $t=0$

$$\Rightarrow v(0)=0;\quad i_L(0)=0$$

WHAT'S DIFFERENT FOR INITIAL ENERGY ?

EXAMPLE 8.10



$$i_L(0) = 29\,mA$$

$$v(0) = 50\,V$$

$$\frac{1}{2RC} = 40{,}000 \qquad \frac{1}{\sqrt{LC}} = 40{,}000$$

∴ CRITICAL DAMPING

⇒ SOL'N OF FORM

$$i_L(t) = I_f + D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

$$\alpha = 40{,}000 \text{ RAD/s}$$

$$(\alpha = 40{,}000 \text{ Nepers/s} \ldots)$$

$I_f = 24\,mA$

WHY ?

INITIAL COND: $i_L(0) = 29\,mA = I_f + D_2$

$\Longrightarrow D_2 = 29\,mA - 24\,mA = 5\,mA$

INITIAL COND: $v(0) = L \left.\dfrac{di_L(t)}{dt}\right|_{t=0}$

WHY?

$$\left.\frac{di_L(t)}{dt}\right|_{t=0} = -\alpha D_2 + D_1$$

$$v(0) = 50 = 25\,mH\,(-40{,}000\,\overset{5mA}{D_2} + D_1\,)$$

$$D_1 = 2{,}200\ A/s$$

$$= 2.2\times10^3\ A/s$$

$$= 2.2\times10^6\ mA/s$$

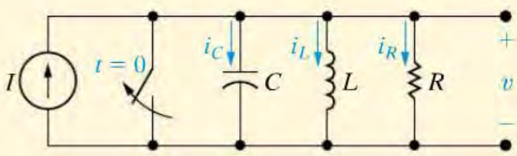
$$i_L(t) = 24 + e^{-40{,}000t}\left(2.2\times10^{6}\,t + 5\right)$$

$$\text{mA} \; ; \; t \geq 0$$

✓ ASSESSMENT PROBLEM

Objective 1—Be able to determine the natural response and the step response of parallel *RLC* circuits

8.6 In the circuit shown, $R = 500\ \Omega$, $L = 0.64$ H, $C = 1\ \mu$F, and $I = -1$ A. The initial voltage drop across the capacitor is 40 V and the initial inductor current is 0.5 A. Find (a) $i_R(0^+)$; (b) $i_C(0^+)$; (c) $di_L(0^+)/dt$; (d) s_1, s_2; (e) $i_L(t)$ for $t \geq 0$; and (f) $v(t)$ for $t \geq 0^+$.



NOTE: Also try Chapter Problems 8.27–8.29.

Answer:
(a) 80 mA;

(b) -1.58 A;

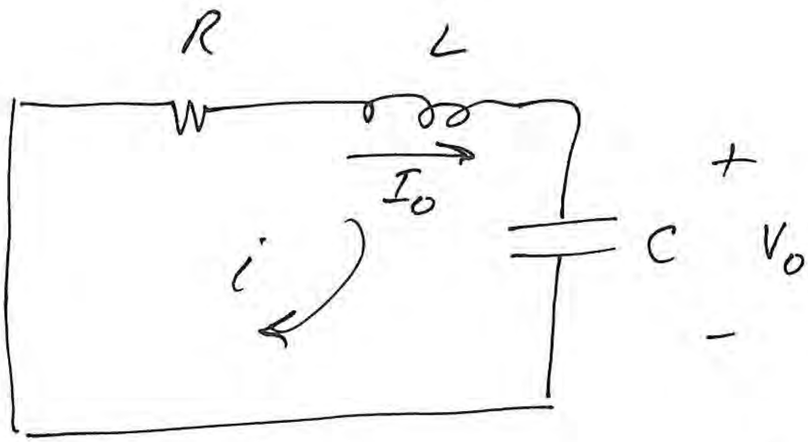
(c) 62.5 A/s;

(d) $(-1000 + j750)$ rad/s, $(-1000 - j750)$ rad/s;

(e) $[-1 + e^{-1000t}[1.5 \cos 750t + 2.0833 \sin 750t]$ A, for $t \geq 0$;

(f) $e^{-1000t}(40 \cos 750t - 2053.33 \sin 750t)$ V, for $t \geq 0^+$.

SERIES RLC CKTS



KVL: $\quad -iR - L\dfrac{di}{dt} - \dfrac{1}{C}\displaystyle\int_0^t i(\tau)d\tau - V_0 = 0$

$\frac{d}{dt}$ BOTH SIDES:

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = 0$$

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

STANDARD APPROACH: HYPOTHESIZE

SOL'N OF FORM Ae^{st}

$$As^2 e^{st} + \frac{R}{L} A s e^{st} + \frac{Ae^{st}}{LC} = 0$$

$$Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0$$

$$s_i = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_i = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

DAMPING FACTOR: $\alpha = \frac{R}{2L}$ RAD/S

(Np/s ?)

RESONANT FREQ, $\omega_0 = \frac{1}{\sqrt{LC}}$ RAD/S

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(OVER)}$$

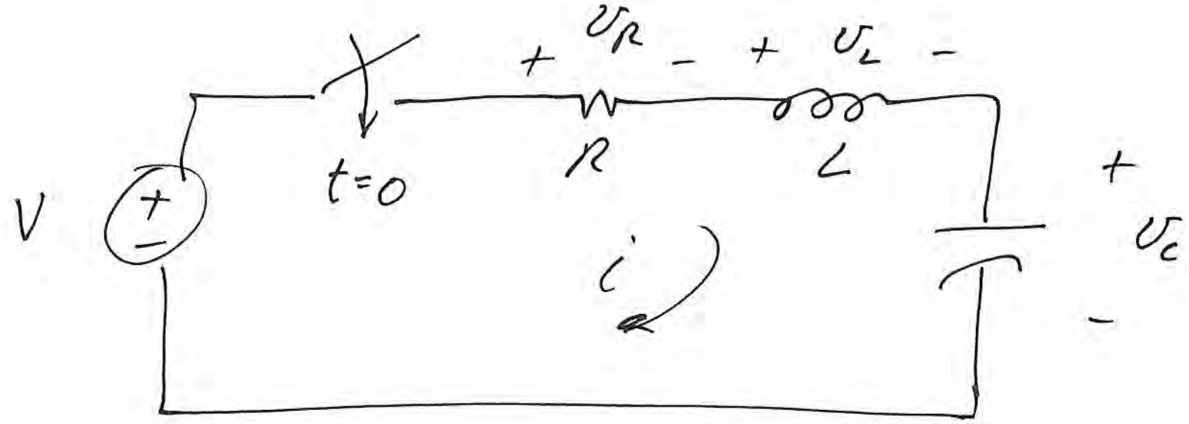
$$i(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

(UNDER)

$$i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t} \quad \text{(CRITICAL)}$$

WHAT TYPE OF RESPONSES ARE THESE ?

STEP RESPONSE



KVL: $V - iR - L\dfrac{di}{dt} - v_c = 0$

RECALL $\quad i = C\dfrac{dv_c}{dt} \Rightarrow \dfrac{di}{dt} = C\dfrac{d^2 v_c}{dt^2}$

$$V = RC\frac{dv_c}{dt} + LC\frac{d^2 v_c}{dt^2} + v_c$$

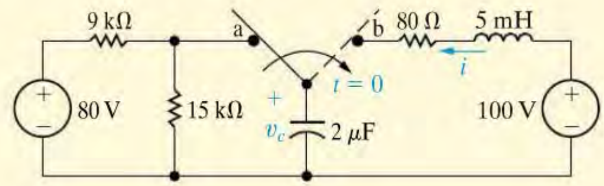
$$\frac{d^2 v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{v_c}{LC} = \frac{V}{LC}$$

WHAT'S DIFFERENT ABOUT
THIS GENERAL FORM?

✓ ASSESSMENT PROBLEMS

Objective 2—Be able to determine the natural response and the step response of series *RLC* circuits

8.7 The switch in the circuit shown has been in position a for a long time. At $t = 0$, it moves to position b. Find (a) $i(0^+)$; (b) $v_C(0^+)$; (c) $di(0^+)/dt$; (d) s_1, s_2; and (e) $i(t)$ for $t \geq 0$.



Answer: (a) 0;

(b) 50 V;

(c) 10,000 A/s;

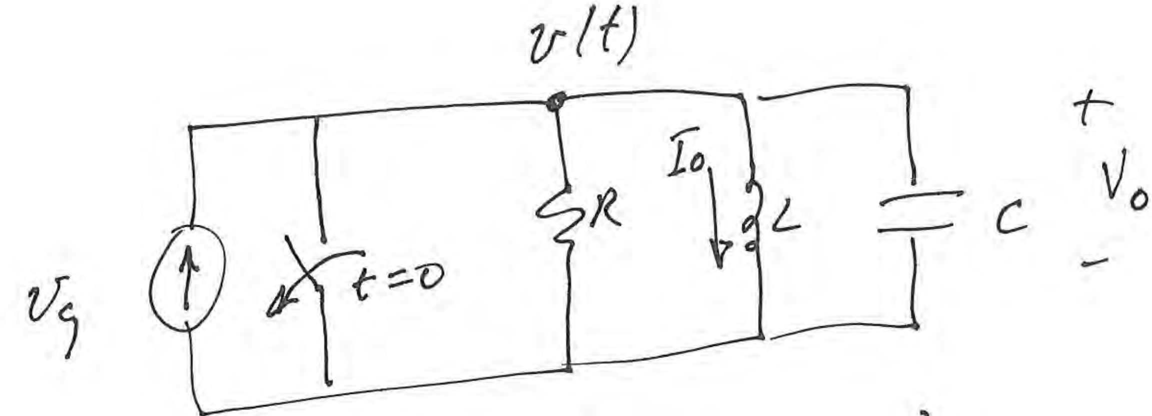
(d) $(-8000 + j6000)$ rad/s, $(-8000 - j6000)$ rad/s;

(e) $(1.67e^{-8000t} \sin 6000t)$ A for $t \geq 0$.

NOTE: Also try Chapter Problems 8.49–8.51.

8.8 Find $v_C(t)$ for $t \geq 0$ for the circuit in Assessment Problem 8.7.

Answer: $[100 - e^{-8000t}(50 \cos 6000t + 66.67 \sin 6000t)]$ V for $t \geq 0$.



TYPE OF RESPONSE?

FORCED + NATURAL

FORM OF NATURAL RESPONSE?

DETERMINE $s_{1,2}$

LOOKING FOR $v(t)$?

$$v(t) = A_1 e^{+s_1 t} + A_2 e^{s_2 t} \quad \text{(e.g.)}$$

B.C.: $v(0) = V_0 = A_1 + A_2$

$$i_c(t) = C \frac{dv(t)}{dt} \; : \; i_c(0) = C \left.\frac{dv(t)}{dt}\right|_{t=0}$$

~~KA~~ KCL: $i_R(0) + i_L(0) + i_c(0) = 0$

$$i_c(0) = -i_L(0) - i_R(0) = -I_0 - V_0/R$$

LOOKING FOR $i_L(t)$?

$$i_L(t) = \underbrace{I_F}_{\text{FORCED}} + \underbrace{D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}}_{\text{NATURAL}} \quad (\text{e.g.})$$

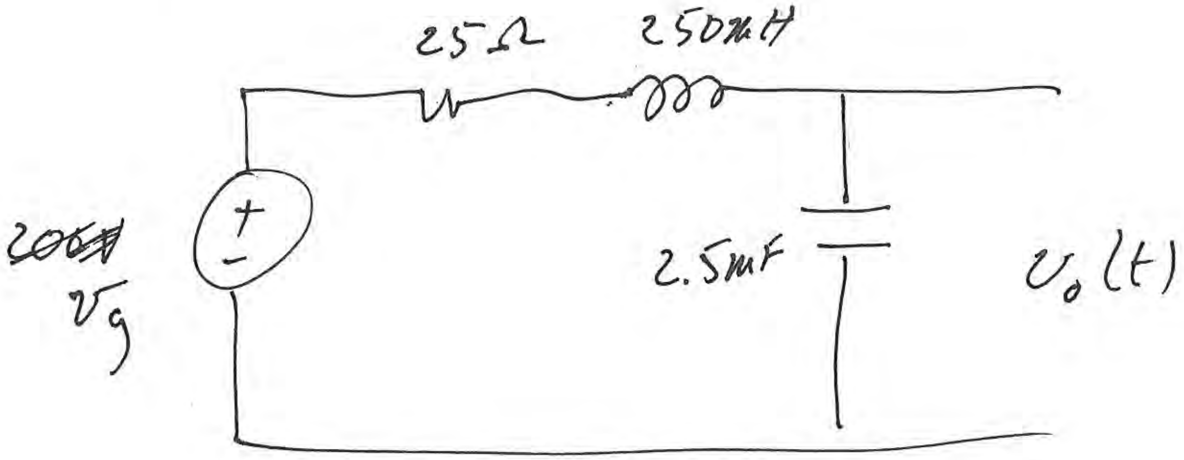
$$\text{B.C.: } i_L(0) = I_F + D_2 = I_0 \rightarrow D_2 = I_0 - I_F$$

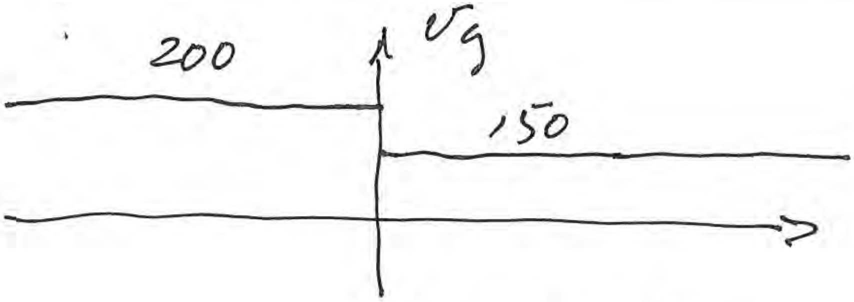
$$v(t) = L \frac{di_L(t)}{dt}$$

$$v(t)\Big|_{t=0} = V_0 = L \frac{di_L(t)}{dt}\Big|_{t=0} = L(D_1 - \alpha D_2)$$

$$D_1 = \frac{V_0}{L} + \alpha D_2$$

8.53





FIND $v_o(t)$

FOR $t < 0$, $v_o(t) = v_o(0) = 200\,V$

FOR $t \to \infty$, $v_o(\infty) \to 150\,V$

$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$

$\alpha = 50 \qquad \omega_0 = 40$

$\alpha > \omega_0$

$\cancel{\omega_0 \geq \alpha}$

$\therefore$ ~~UNDERDAMPED~~

OVERDAMPED

$\cancel{\omega_d = \sqrt{\omega_0^2 - \alpha}}$

$$s_{1,2} = -50 \pm \sqrt{(50)^2 - (40)^2}$$

$$= -50 \pm 30$$

$$= -80, -20$$

$$v_o(t) = 150V + A_1 e^{-80t} + A_2 e^{-20t}$$

$$v_o(0) = 200 = 150 + A_1 + A_2$$

$$\boxed{A_1 + A_2 = 50}$$

$$i(t) = C \frac{dv_o(t)}{dt}$$

$$i(0) = C \frac{dv_o(t)}{dt}\bigg|_{t=0} = 0$$

BECAUSE LOOP CURRENT AT $t=0$ IS 0 (INDUCTOR CURRENT CANNOT CHANGE INSTANTANEOUSLY)

$$C \frac{dv_o}{dt}\bigg|_{t=0} = C\left(-80A_1 e^{-80t} - 20A_2 e^{-20t}\right)\bigg|_{t=0} = 0$$

$$\boxed{-80A_1 - 20A_2 = 0}$$

$\cancel{-80A_1}$

$$A_1 + A_2 = 50$$
$$-4A_1 - A_2 = 0$$

$$-3A_1 = 50 \longrightarrow A_1 = -\frac{50}{3}$$

$$A_2 = 50 - A_1 = 50 + \frac{50}{3}$$

$$= \frac{150+50}{3} = \frac{200}{3}$$

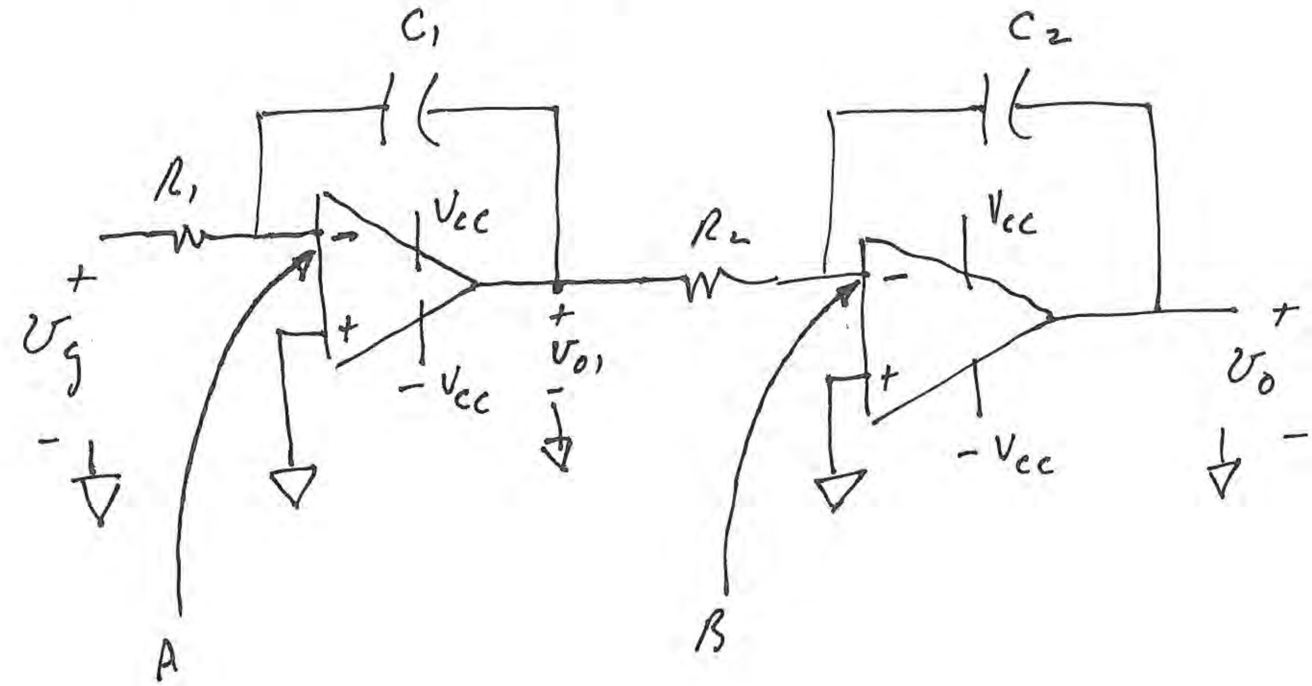
$$v_0(t) = 150 - \frac{50}{3}e^{-80t} + \frac{200}{3}e^{-20t}$$

check $v_0(t)\big|_{t=0} = 150 - \frac{50}{3} + \frac{200}{3}$

$$= 150 + \frac{150}{3} = \frac{450+150}{3} = \frac{600}{3}$$

$$= 200 \checkmark$$

8.5 TWO OP-AMPS



WANT: RELATIONSHIP BETWEEN v_o & v_g

KCL AT NODE A:

WHY?

$$\frac{(0-v_g)}{R_1} + C_1 \frac{d}{dt}(0-v_{o1}) = 0$$

$$\frac{dv_{o1}}{dt} = -\frac{1}{R_1 C_1} v_g$$

WHAT DOES THIS OP-AMP DO?

KCL AT NODE B:

$$\frac{(0-v_{01})}{R_2} + C_2 \frac{d}{dt}(0-v_0) = 0$$

$\therefore$

$$\frac{dv_0}{dt} = -\frac{1}{R_2C_2} v_{01}$$

$\frac{d}{dt}$ BOTH SIDES: $\frac{d^2v_0}{dt^2} = -\frac{1}{R_2C_2}\frac{dv_{01}}{dt}$

RECALL EQ FOR FIRST OP-AMP:

$$\frac{dv_{01}}{dt} = -\frac{1}{R_1C_1} v_g$$

TOGETHER: $\frac{d^2v_0}{dt^2} = \frac{1}{R_1C_1}\frac{1}{R_2C_2} v_g$

EXAMPLE 8.13

$R_1 = 250\,k\Omega$ $\qquad R_2 = 500\,k\Omega$ $\qquad V_{CC1} = 5V$

$C_1 = 0.1\,\mu F$ $\qquad C_2 = 1\,\mu F$ $\qquad V_{CC2} = 9V$

NO INITIAL ENERGY STORED

$$v_g = \begin{cases} 0 \,; & t = 0^- \\ 25mV \,; & t = 0^+ \end{cases}$$

a) FIND $v_o(t)$; $0 \le t \le t_{SAT}$

b) FIND TIME TO SATURATION

$$\frac{d^2 v_o}{dt^2} = \underbrace{\frac{1}{R_1 C_1}}_{=40} \overbrace{\frac{1}{R_2 C_2}}^{=2} \, v_g = 25mV;\ t > 0$$

$$\frac{d^2 v_o}{dt^2} = 2 \quad V/s^2 ;\ t > 0$$

HOW TO SOLVE ?

$$\frac{d^2 v_0}{dt^2} = 2 \ V/s^2$$

INTEGRATE: $\frac{dv_0}{dt} = 2t + A_1 \quad V/s$

INTEGRATE: $v_0 = t^2 + A_1 t + A_2 \quad V$

CONSTANTS FROM INITIAL COND'S.

WHAT DO WE KNOW?

VOLTAGE ACROSS A CAP CANNOT CHANGE INSTANTANEOUSLY

$$\Rightarrow \left.\frac{dv_0}{dt}\right|_{t=0} = 0 \quad \Rightarrow \left.(2t + A_1)\right|_{t=0} = 0$$

$$\therefore A_1 = 0$$

SO WHAT IS A_2?

$$v_o(t) = t^2 + v_o(0) \ \text{V}$$

BECAUSE NO INITIAL STORED ENERGY,

$$v_o(0) = 0$$

NOTE NUMERICAL CONSTANT

$$\boxed{v_o(t) = t^2 \text{ V}; \ 0 \le t \le t_{SAT}}$$

SATURATION WHEN $v_o(t) = V_{CC2} = 9\text{V}$

$$\Rightarrow t_{SAT} = 3\text{s}$$

CAN 1ST OP-AMP SATURATE SOONER?

RECALL FOR 1ST STAGE

$$\frac{dv_{o1}}{dt} = -\underbrace{\frac{1}{R_1 C_1}}_{=40} v_S = 25\text{mV}; \ t>0$$

$$\frac{dv_{o1}}{dt} = -1 \ \text{V/s}$$

$$\frac{dv_{01}}{dt} = -1\ V/s$$

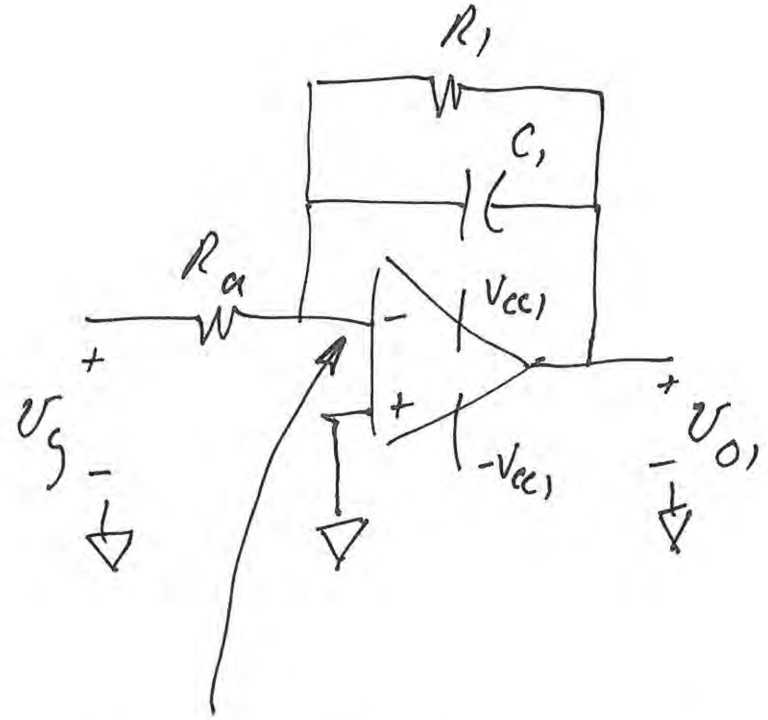
INTEGRATE: $v_{01}(t) = -1\,t + \overset{\approx 0}{v_{01}(0)}$ V

SECOND OP-AMP SATURATES AT $t = 3$ s

AT THIS POINT $v_{01}(t_{SAT}) = -3$ V

(WITHIN $\pm V_{CC1}$ $\therefore$ OK)

INTEGRATING AMP W FEEDBACK RESISTORS

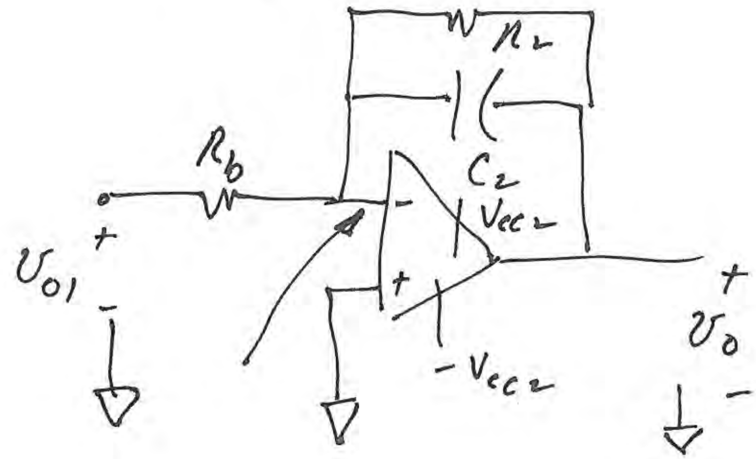


KCL: $$\frac{(0-v_g)}{R_a} + \frac{(0-v_{o1})}{R_1} + C_1\frac{d}{dt}(0-v_{o1}) = 0$$

$$\frac{dv_{o1}}{dt} + \frac{1}{R_1C_1}v_{o1} = -\frac{v_g}{R_aC_1}$$

$(\tau_1 = R_1C_1)$ $$\frac{dv_{o1}}{dt} + \frac{v_{o1}}{\tau_1} = -\frac{v_g}{R_aC_1} \qquad \text{I}$$

SECOND STAGE



SAME FORM AS PREVIOUS:

$$\frac{dv_o}{dt} + \frac{v_o}{\tau_2} = -\frac{v_{01}}{R_b C_2} \qquad \text{(II)}$$

$\frac{d}{dt}$ BOTH SIDES:

$$\frac{d^2 v_o}{dt^2} + \frac{1}{\tau_2}\frac{dv_o}{dt} = -\frac{1}{R_b C_2}\frac{dv_{01}}{dt} \qquad \text{(II')}$$

WITH (I), (I'), (II) & (II') ELIMINATE dv_{01}/dt & v_{01}

$$\frac{d^2 v_o}{dt^2} + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\frac{dv_o}{dt} + \frac{v_o}{\tau_1 \tau_2} = \frac{v_g}{R_a C_1 R_b C_2}$$

DETAILS ON PREVIOUS PAGE

$$\frac{dv_{01}}{dt} + \frac{v_{01}}{\tau_1} = -\frac{v_g}{R_a C_1} \qquad \text{(I)}$$

$$\frac{dv_0}{dt} + \frac{v_0}{\tau_2} = -\frac{v_{01}}{R_b C_2} \qquad \text{(II)}$$

$$\frac{d^2 v_0}{dt^2} + \frac{1}{\tau_2}\frac{dv_0}{dt} = -\frac{1}{R_b C_2}\frac{dv_{01}}{dt} \qquad \text{(II)}'$$

COMBINE (I) & (II)', SOLVING FOR $\frac{dv_{01}}{dt}$

$$\frac{dv_{01}}{dt} + \frac{v_{01}}{\tau_1} = -\frac{v_g}{R_a C_1} \qquad -R_b C_2\left(\frac{dv_0}{dt} + \frac{v_0}{\tau_2}\right)$$

$$\frac{dv_{01}}{dt} = \frac{R_b C_2}{\tau_1}\left(\frac{dv_0}{dt} + \frac{v_0}{\tau_2}\right) - \frac{v_g}{R_a C_1}$$

SUB INTO RHS OF (II)'

$$\frac{d^2 v_0}{dt^2} + \frac{1}{\tau_2}\frac{dv_0}{dt} = -\frac{1}{R_b C_2}\left[\frac{R_b C_2}{\tau_1}\left(\frac{dv_0}{dt} + \frac{v_0}{\tau_2}\right) - \frac{v_g}{R_a C_1}\right]$$

$$\frac{d^2 v_0}{dt^2} + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\frac{dv_0}{dt} + \frac{v_0}{\tau_1 \tau_2} = \frac{v_g}{R_a R_b C_1 C_2}$$

USE ESTABLISHED SOLUTION APPROACH

HYPOTHESIZE SOL'N. OF FORM

$$v_o = Ae^{st}$$

D.E. BECOMES (HOMOGENEOUS FORM)

$$As^2e^{st} + As\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)e^{st} + \frac{1}{\tau_1\tau_2}Ae^{st} = 0$$

$$Ae^{st}\Big[\underbrace{s^2 + s\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right) + \frac{1}{\tau_1\tau_2}}_{\text{CHARACTERISTIC EQ.}}\Big] = 0$$

ROOTS: $s_1 = -1/\tau_1$; $s_2 = -1/\tau_2$

$$v_o = A_1 e^{-t/\tau_1} + A_2 e^{-t/\tau_2} \text{ (IF DISTINCT)}$$

$$v_o = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t} \text{ (IF NOT DISTINCT)}$$

EXAMPLE 8.14

$R_a = 100\,k\Omega$ $\qquad$ $R_b = 25\,k\Omega$

$R_1 = 500\,k\Omega$ $\qquad$ $R_2 = 100\,k\Omega$

$C_1 = 0.1\,\mu F$ $\qquad$ $C_2 = 1\,\mu F$

$$V_{cci} = \pm 6\,V$$

$$v_g = \begin{cases} 0 \ ; & t = 0^- \\ 250\,mV \ ; & t = 0^+ \end{cases}$$

NO STORED ENERGY AT $t<0$

$$\frac{d^2 v_o}{dt^2} + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\frac{dv_o}{dt} + \frac{v_o}{\tau_1 \tau_2} = \frac{v_g}{R_a C_1 R_b C_2}$$

$$\frac{d^2 v_o}{dt^2} + (20 + 10)\frac{dv_o}{dt} + (20)(10)\, v_o = 1000 \ V/s^2$$

$$s_i = -1/\tau_i \ ; \quad s_1 = -20 \text{ RAD/s}$$

$$s_2 = -10 \text{ RAD/s}$$

$$\frac{d^2 v_o}{dt^2} + 30\frac{dv_o}{dt} + 200\, v_o = 1000$$

AS $t \to \infty$ $\quad v_o(t) \to v_o(\infty)$ (CONSTANT)

$$\Rightarrow \left.\frac{d}{dt} v_o(t)\right|_{t\to\infty} = 0$$

$$\therefore \quad 200\, v_o(\infty) = 1000 \longrightarrow \overset{v_o(\infty)}{v_o} = 5\text{ V}$$

$$v_o(t) = 5 + A_1 e^{-20t} + A_2 e^{-10t}$$

INITIAL CONDS: $\quad v_o(0) = 5 + A_1 + A_2 = 0$

$$\left.\frac{dv_o}{dt}\right|_{t=0} = -20A_1 - 10A_2 = 0$$

INITIAL COND'S. GIVE

$$A_1 = 5\text{ V}, \quad A_2 = -10\text{ V}$$

$$v_o(t) = 5\left(1 + e^{-20t}\right) - 10e^{-10t} \quad V;\ t \geq 0$$

RECALL D.E. FOR 1ST STAGE

$$\frac{dv_{o1}}{dt} + \frac{v_{o1}}{\tau_1} = -\frac{v_s}{R_a C_1}$$

$$\frac{dv_{o1}}{dt} + v_{o1}20 = 25\ V/s$$

SOLUTION?

STANDARD APPROACH FOR HOMOGENEOUS D.E.

$$v_{o1} = Ae^{st}$$

$$Ase^{st} + 20Ae^{st} = 0$$

$$Ae^{st}(s + 20) = 0 \implies s = -20\ RAD/s$$

FURTHER, AS $t \to \infty$ $v_{01}(t) \to v_{01}(\infty)$ (CONST)

$$\Rightarrow \left.\frac{dv_{01}}{dt}\right|_{t\to\infty} = 0$$

FOR $t \to \infty$, D.E. IS $20\, v_{01}(\infty) = -25$

$$v_{01}(\infty) = -1.25\,V$$

$$v_{01}(t) = Ae^{-20t} - 1.25\,V$$

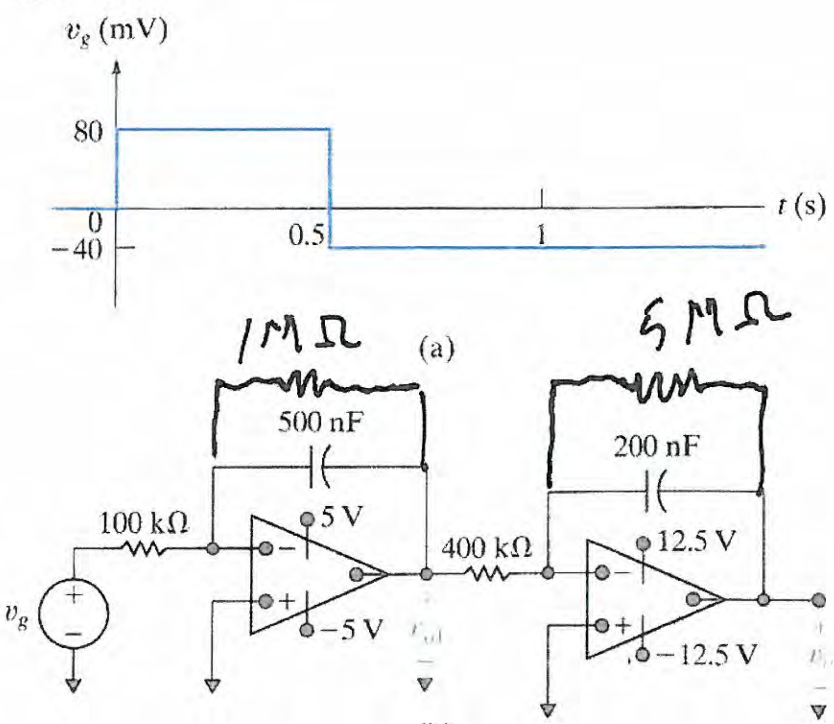
KNOW $v_{01}(0) = 0 = A - 1.25$

$$\therefore\ v_{01}(t) = 1.25\left(e^{-20t} - 1\right) V;\ t \geq 0$$

8.64 The circuit in Fig. P8.63(b) is modified by adding a 1 MΩ resistor in parallel with the 500 nF capacitor and a 5 MΩ resistor in parallel with the 200 nF capacitor. As in Problem 8.63, there is no energy stored in the capacitors at the time the signal is applied. Derive the numerical expressions for $v_o(t)$ and $v_{o1}(t)$ for the time intervals $0 \le t \le 0.5$ s and $t \ge 0.5$ s.

PSPICE
MULTISIM

Figure P8.63



ESTABLISH D.E.

$$\frac{d^2 v_0}{dt^2} + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\frac{dv_0}{dt} + \frac{v_0}{\tau_1 \tau_2} = \frac{v_g}{R_a C_1 R_b C_2}$$

CALCULATE CONSTANTS

DETERMINE STEADY STATE SOL'N

DETERMINE NATURAL RESPONSE

APPLY INITIAL COND'S.

$\left(v_0(0), \; \& \; \frac{dv_0(0)}{dt} \right)$

~~⊗$\ast$~~ SOL'N FOR $t \geq 0$

DETERMINE 2^{ND} FORCING FCN

AT TIME t_{sw}

FIRST STEP SOL'N AT t_{sw} FOR

INITIAL CONDS.

$\Rightarrow$ SOL'N FOR $t > t_{sw}$

$$\tau_1 = (1M\Omega)(500nF) = 0.5 \text{ SEC}$$

$$\tau_2 = (5M\Omega)(200nF) = 1.0 \text{ SEC}$$

$$R_a = 100k\Omega$$

$$R_b = 400k\Omega$$

$$C_1 = 500nF$$

$$C_2 = 200nF$$

$$R_a C_1 R_b C_2 = 0.004$$

D.E. IS

$$\frac{d^2 v_o}{dt^2} + 3\frac{dv_o}{dt} + 2v_o = 250\,v_g$$

STEADY STATE SOLUTION IS SUCH THAT

$$\frac{v_o(\infty)}{\tau_1 \tau_2} = \frac{v_g}{R_a C_1 R_b C_2}$$

$$\text{OR} \quad v_o(\infty) = \frac{R_1 C_1 R_2 C_2}{R_a C_1 R_b C_2} v_g$$

$$v_0(\infty) = \frac{R_1 R_2}{R_a R_b} v_g = 125 v_g$$

COMPLETE SOL'N =

STEADY STATE SOL'N + NATURAL RESP

NATURAL RESPONSE IS SOLUTION TO HOMOG EQ

$$\frac{d^2 v_0}{dt^2} + 3\frac{dv_0}{dt} + 2v_0 = 0$$

HYPOTHESIZE SOL'N $v_0(t) = Ae^{st}$

D.E. BECOMES

$$As^2 e^{st} + 3Ase^{st} + 2Ae^{st} = 0$$

$$Ae^{st}(s^2 + 3s + 2) = 0$$

$$s_i = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm 1}{2}$$

$$s_i = -2, -1 \qquad \text{(OVERDAMPED)}$$

$$v_0(t) = v_F + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

— DIGRESSION —

WE ALSO NEED $v_{01}(t)$:

$$\frac{dv_{01}}{dt} + \frac{v_{01}}{\tau_1} = -\frac{v_g}{R_a C_1}$$

STEADY STATE SOL'N SUCH THAT

$$\frac{v_{01}(\infty)}{\tau_1} = -\frac{v_g}{R_a C_1} \longrightarrow v_{01}(\infty) = -\frac{R_1}{R_a} v_g$$

$$v_{01}(\infty) = -10 v_g$$

SOLUTION FORM IS

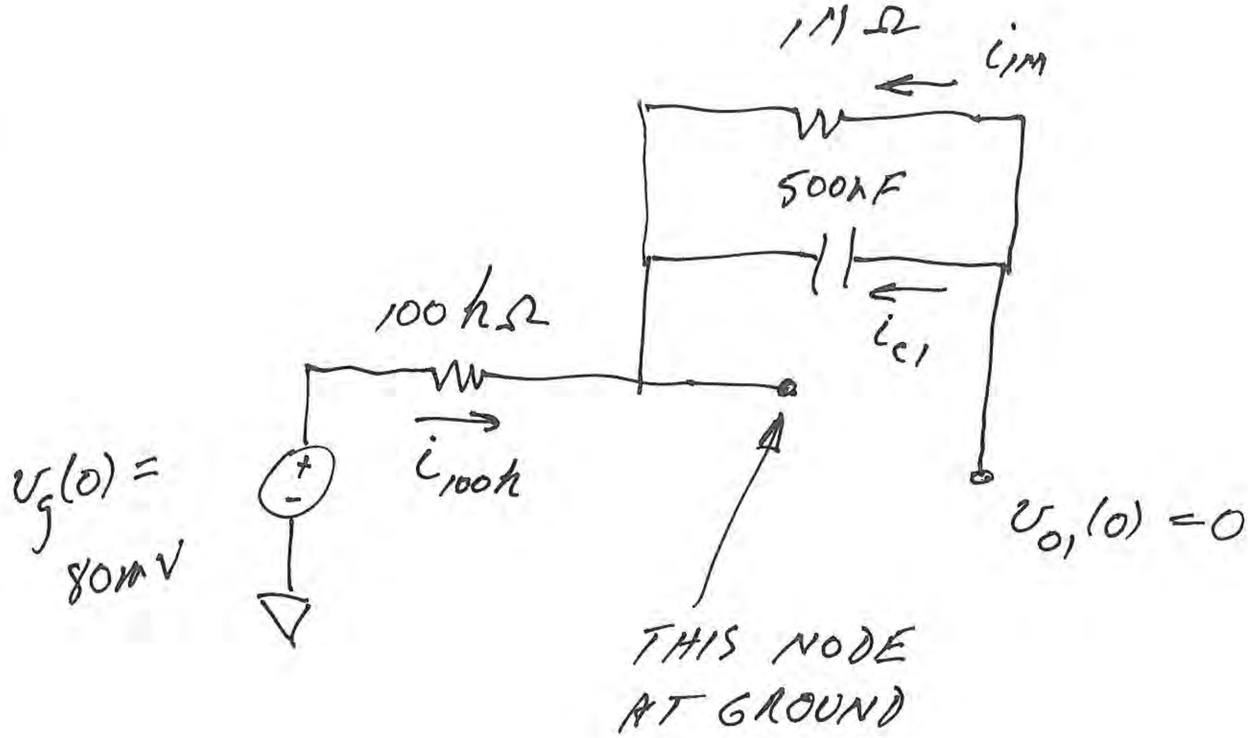
$$v_{01}(t) = v_{01}(\infty) + A e^{-\tau_1 t}$$

INITIAL CONDITION IS $v_{01}(0) = 0$

(VOLTAGE ACROSS CAP.
CANNOT CHANGE INSTANTANEOUSLY)

$$v_{01}(t) = -10v_g + 10v_g e^{-2t}$$

$$\frac{dv_{01}(t)}{dt} = -20v_g e^{-2t}$$



OBVIOUSLY $i_{1M} = \dfrac{0V}{1M\Omega} = 0$

$$i_{c1}(0) = C_1 \left.\frac{dv_{01}(t)}{dt}\right|_{t=0}$$

$$= (500nF)(-20)(80mV) = -0.8\mu A$$

$$i_{100k}(0) = \frac{80mV}{100k\Omega} = 0.8\mu A$$

CURRENTS CHECK ✓

— END DIGRESSION —

$$v_0(t) = 125 v_g + A_1 e^{-2t} + A_2 e^{-t}$$

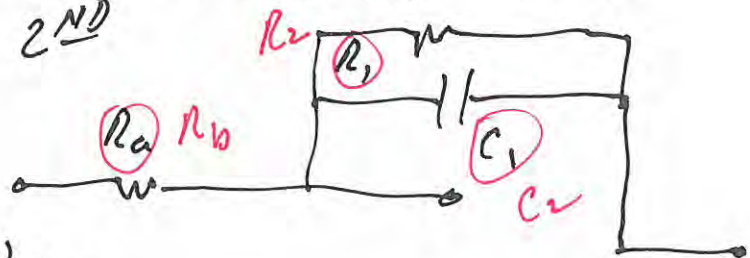
INITIAL COND:

① $v_0(0) = 125 v_g + A_1 + A_2 = 0$ (NO INITIAL ENERGY, VOLTAGE ACROSS CAPACITOR CANNOT CHANGE INSTANTANEOUSLY)

WE NEED ANOTHER EQUATION FOR SECOND UNKNOWN: HAS TO BE

$$C_2 \frac{dv_0(t)}{dt}\bigg|_{t=0}$$

INSPECT ~~1ST~~ 2ND STAGE AT $t=0$



$v_{01}(0) = 0$

$v_0(0) = 0$

NO CURRENT FLOWS THROUGH FEEDBACK CKT AT $t=0$

$$\therefore C_2 \frac{dv_0(t)}{dt}\bigg|_{t=0} = 0$$

$$\left.\frac{dv_o(t)}{dt}\right|_{t=0} = -2A_1 e^{-2t} - A_2 e^{-t}\Big|_{t=0}$$

(2) $$-2A_1 - A_2 = 0$$

WITH EQ'S (1) & (2),

$$A_1 = 125 v_S$$

$$A_2 = -250 v_S$$

$$\boxed{\begin{aligned} v_o(t) &= 125 v_S + 125 v_S e^{-2t} - 250 v_S e^{-t}; \quad t \geq 0 \\ v_{o1}(t) &= -10 v_S + 10 v_S e^{-2t}; \quad t \geq 0 \end{aligned}}$$

NOW FOR $0 \leq t \leq 0.5S$, $v_S = 80mV$

$$v_o(t) = 10 + 10e^{-2t} - 20e^{-t}\ V; \quad 0 \leq t \leq 0.5S$$

$$v_{o1}(t) = -0.8 + 0.8e^{-2t}\ V; \quad 0 \leq t \leq 0.5S$$

AT $t=0.5$, THE SOURCE IS SWITCHED TO

$$v_S = -40\,mV$$

THE ABOVE SOLUTIONS EVALUATED AT $t=0.5$ BECOME THE INITIAL CONDITIONS FOR THE SOLUTIONS OF THE SAME FORM

$$v_o(t)\Big|_{t=0.5} = 10 + 10e^{-1} - 20e^{-0.5}$$

$$= 1.5482\,V$$

$$v_o(t) = v_o(\infty) + A_1 e^{-2(t-0.5)} + A_2 e^{-(t-0.5)}$$

RECALL $v_o(\infty) = 125\, v_S$

NOW $v_S = -40\,mV \Rightarrow v_o(\infty) = -5\,V$

INITIAL CONDITION IS

3

$$v_o(t)\Big|_{t=0.5} = -5 + A_1 + A_2 = 1.5482\,V$$

AGAIN, WE NEED A $2^{\underline{ND}}$ EQ :

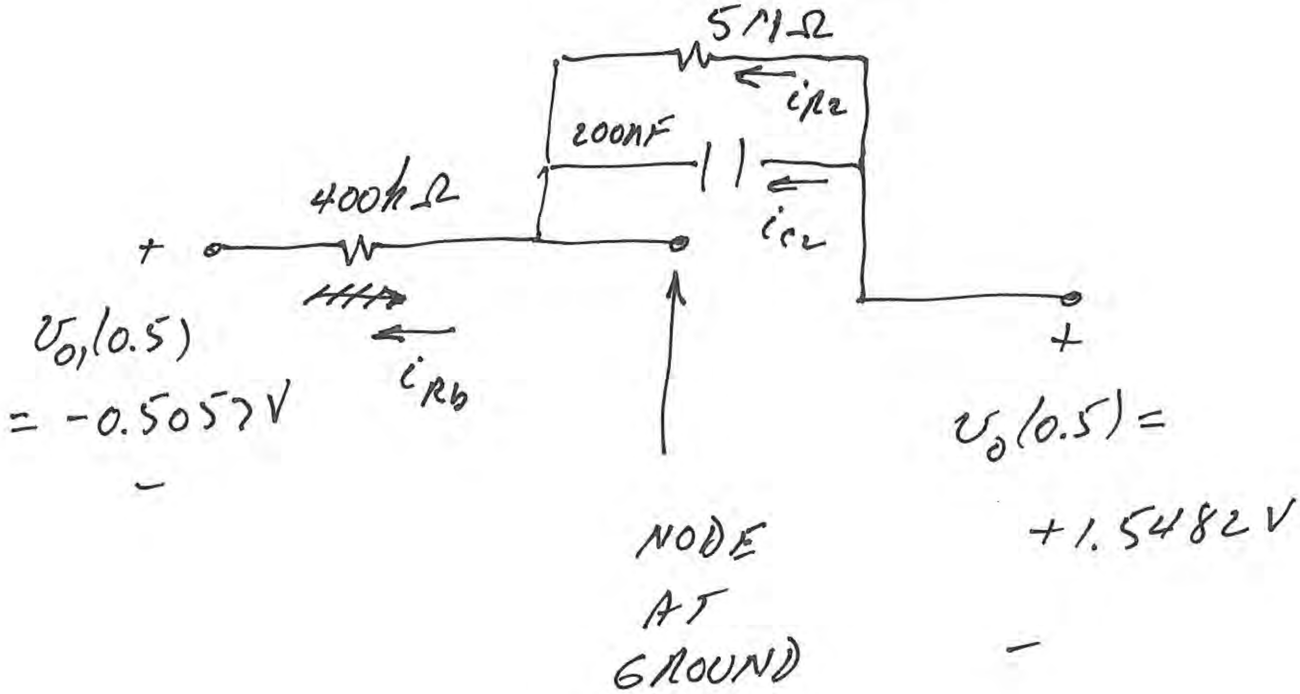
$$C_2 \frac{dv_o(t)}{dt}\Big|_{t=0.5} = C_2(-2A_1 - A_2) = i_{C2}(0.5)$$

NEED TO DETERMINE i_{C_2} AT $t = 0.5$

INSPECT $2^{\underline{ND}}$ STAGE AT $t = 0.5$

NOTE $v_{o_1}(0.5) = -0.8 + 0.8e^{-1}$

$= -0.5057\ V$



$$i_{R_2}(0.5) = \frac{1.5482\ V}{5M\Omega} = 0.30964\ \mu A$$

$$i_{Rb}(0.5) = \frac{0.5057\ V}{400k\Omega} = 1.2643\ \mu A$$

$$i_{C_2}(0.5) = i_{Rb}(0) - i_{R_2}(0.5) = 0.9546\ \mu A$$

2ND EQUATION IS

$$C_2(-2A_1 - A_2) = i_{C2}(0.5)$$

$$-2A_1 - A_2 = \frac{0.9546\mu A}{200uF}$$

(4)

$$-2A_1 - A_2 = 4.7731$$

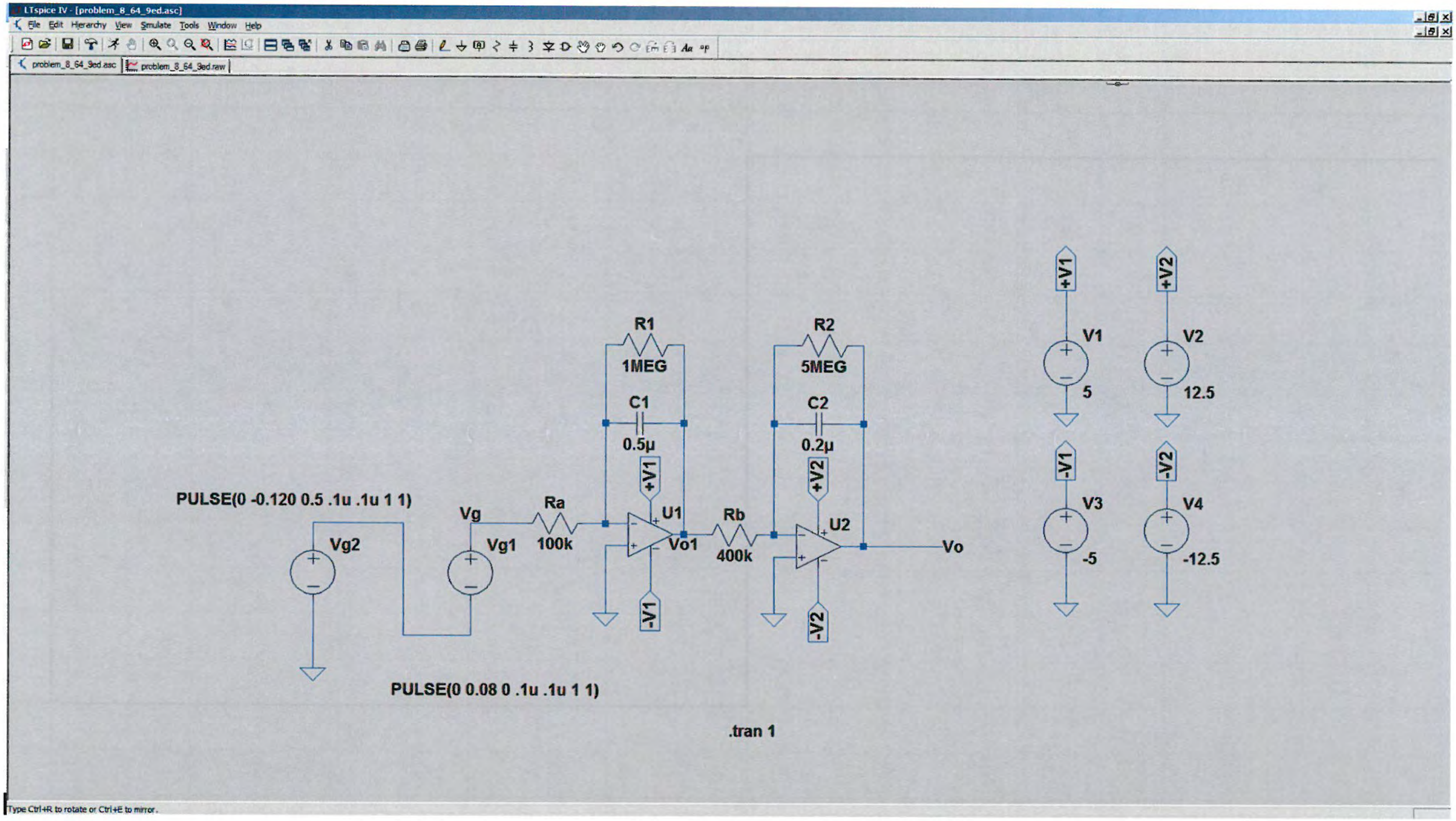
EQUATIONS (3) & (4) GIVE

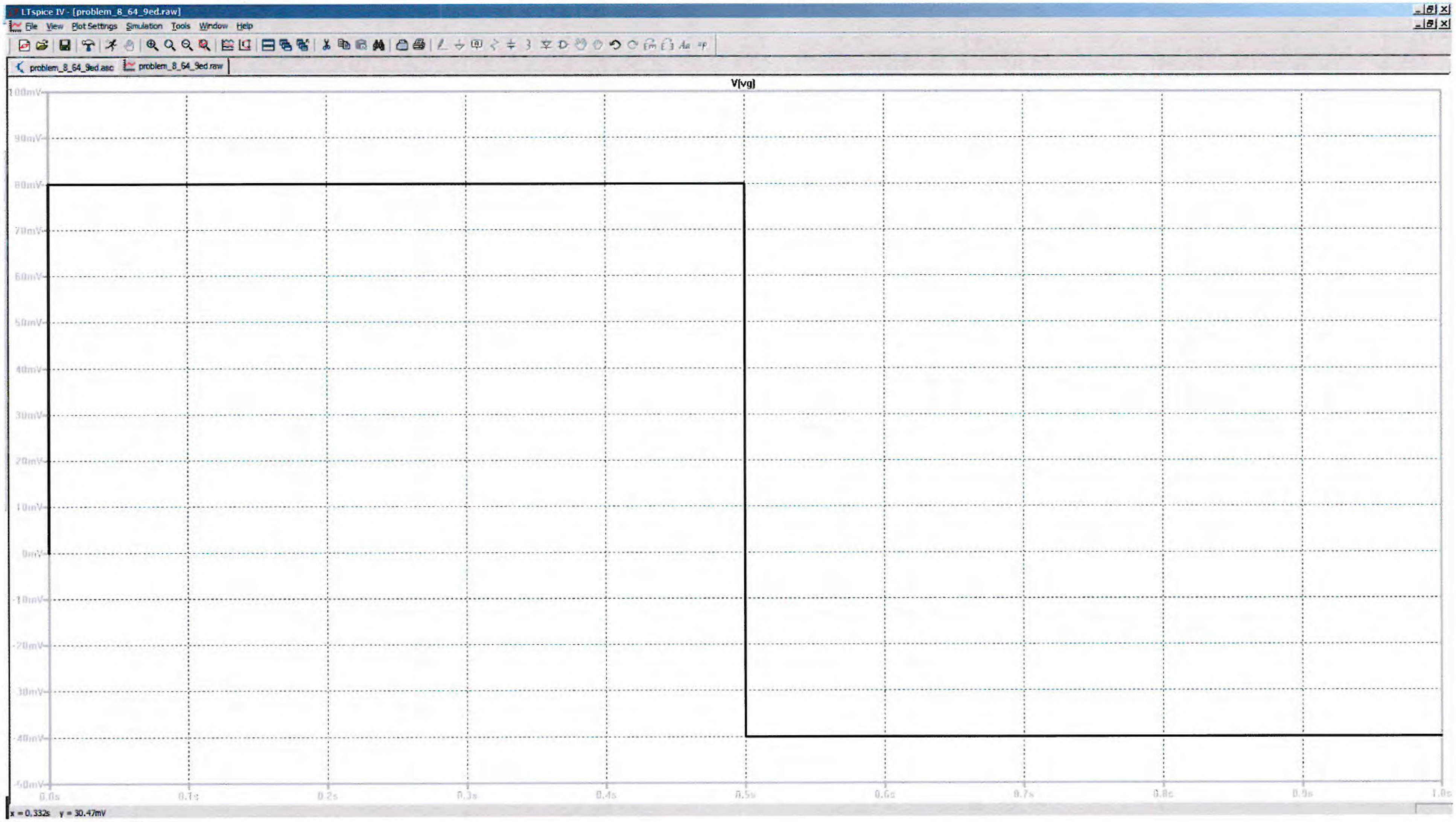
$$A_1 = -11.3211 V$$

$$A_2 = 17.8691 V$$

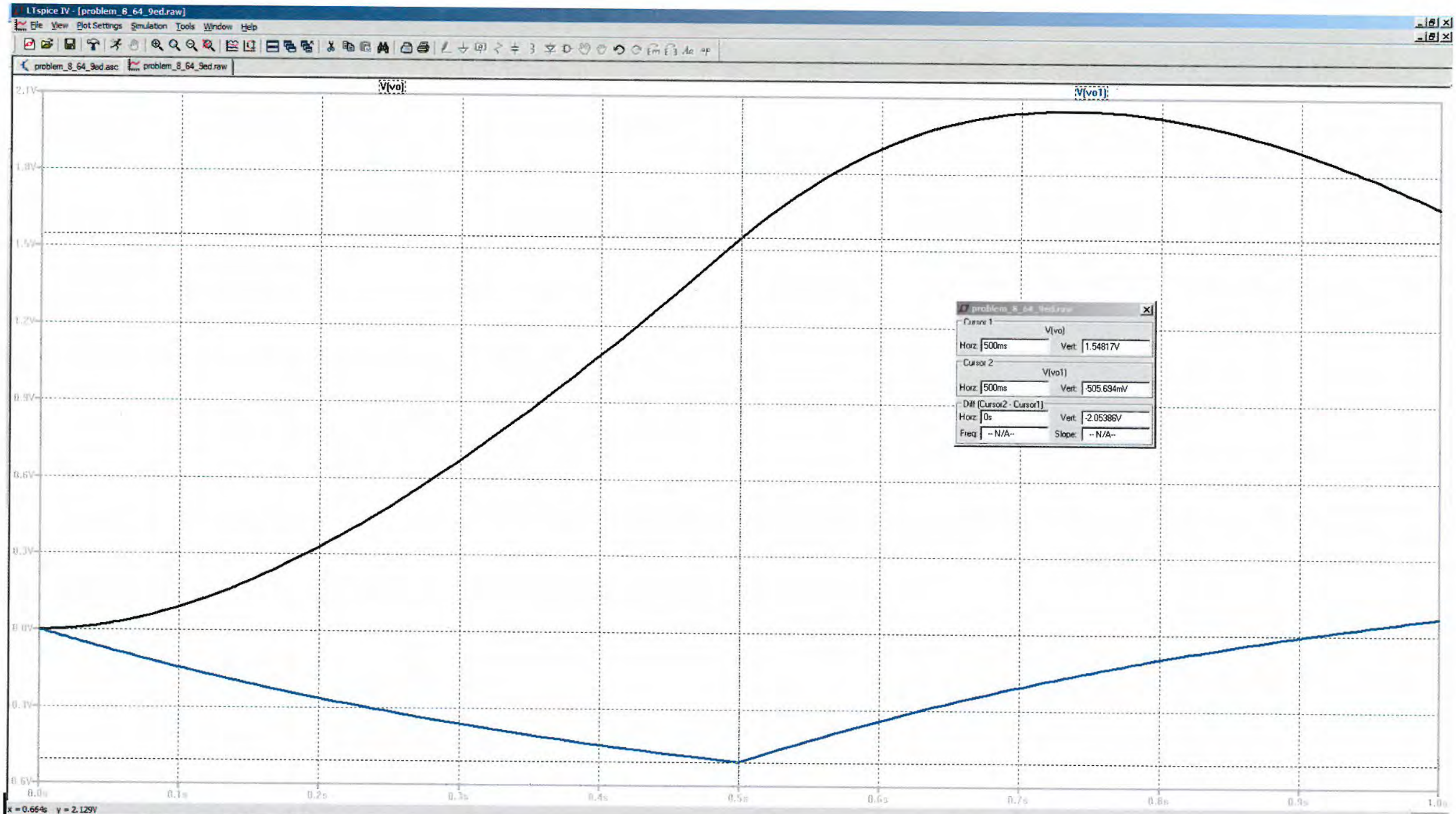
$$\boxed{v_o(t) = -5 - 11.3211 e^{-2(t-0.5)} + 17.8691 e^{-(t-0.5)} \; V \qquad 0.5 \le t < \infty}$$

Problem 8.64, 9th edition

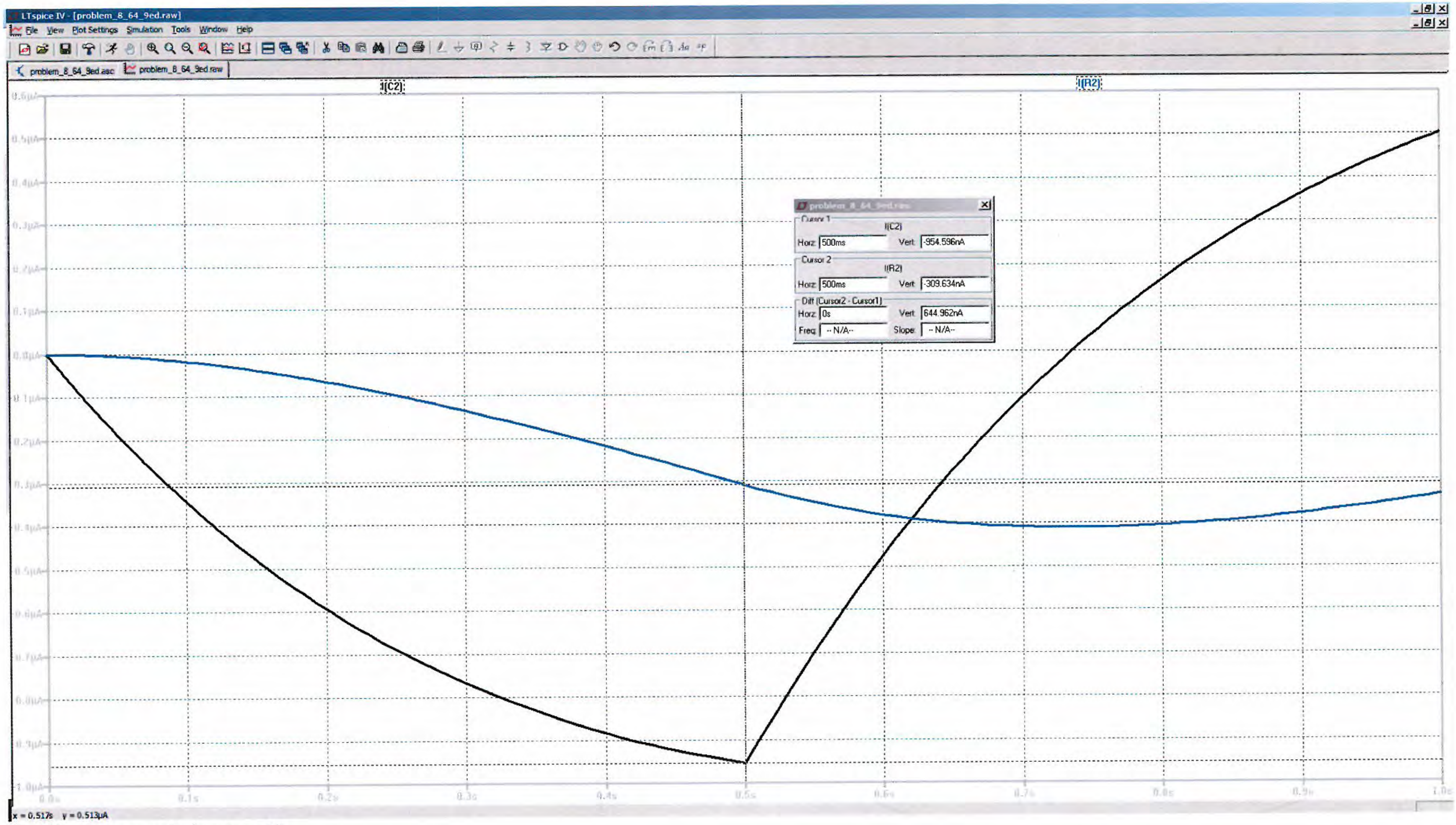




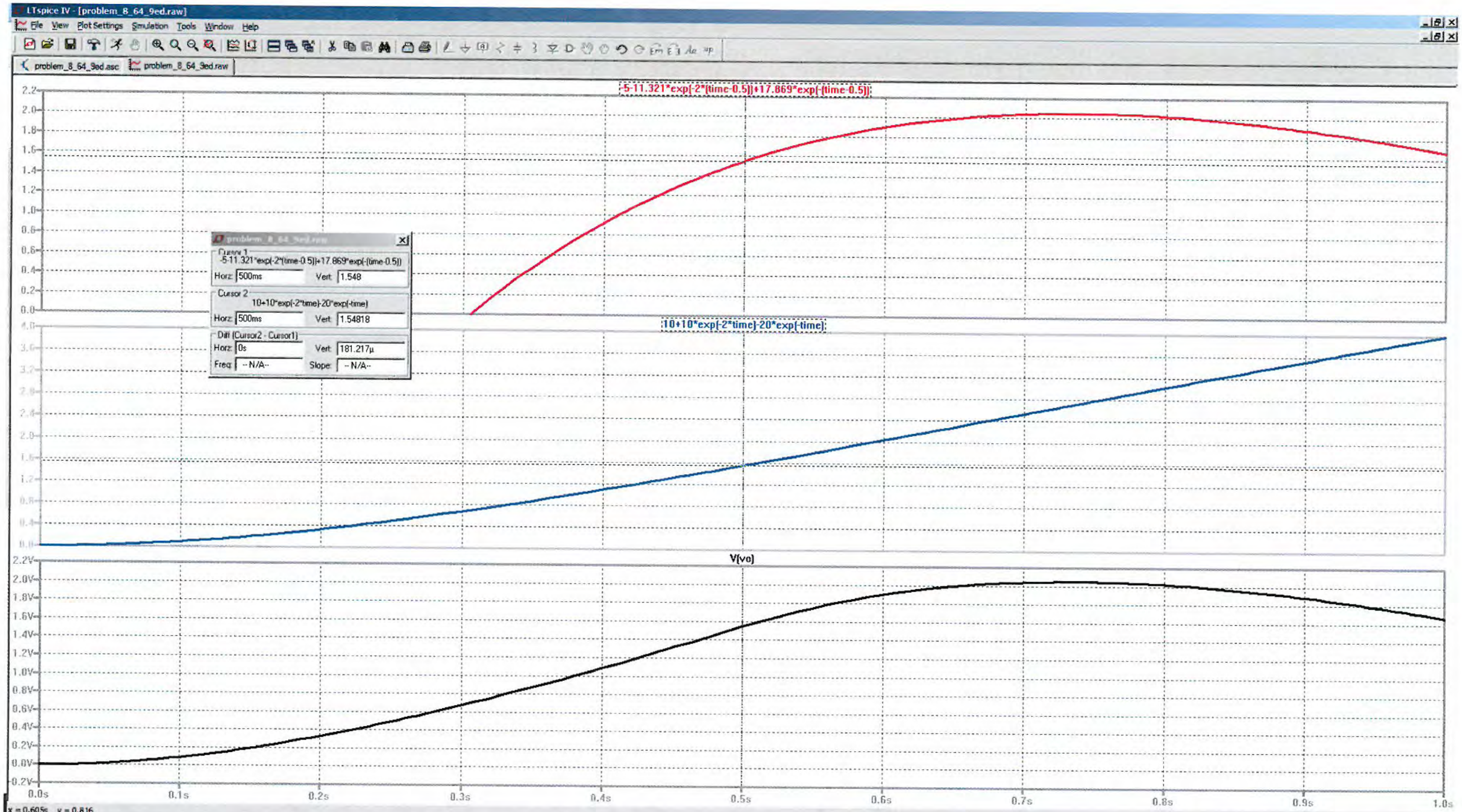
Source waveform



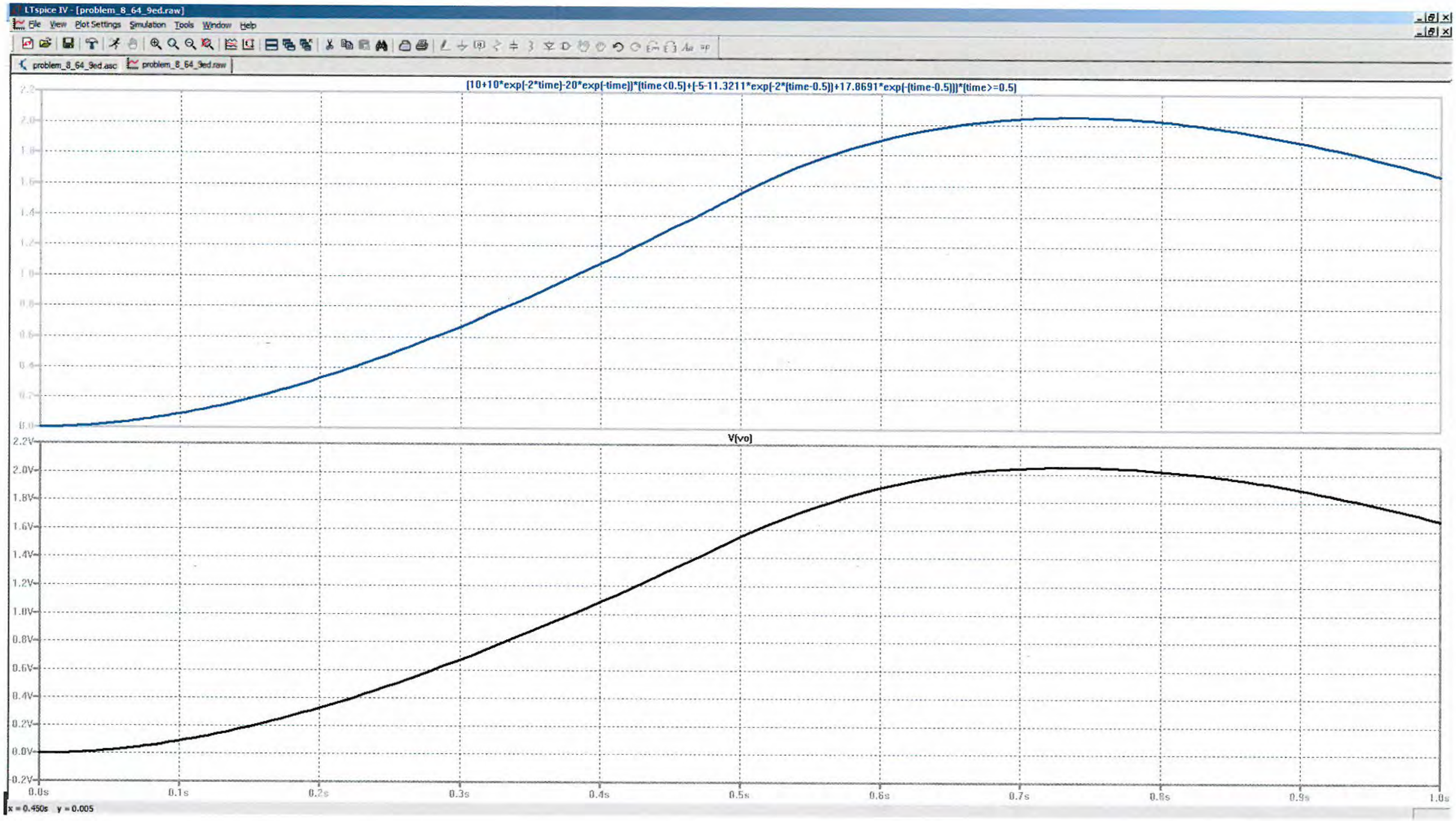
Total solution for stages 1 and 2



Feedback currents in stage 2



Total solution (bottom), solution for t > 0.5 (top) and for 0 < t < 0.5 (middle)



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problem_8_64_9ed.asc
problem_8_64_9ed.raw
(10+10*exp(-2*time)-20*exp(-time))*(time<0.5)+(-5-11.3211*exp(-2*(time-0.5))+17.8691*exp(-(time-0.5)))*(time>=0.5)
V(vo)
2.2V
2.0V
1.8V
1.6V
1.4V
1.2V
1.0V
0.8V
0.6V
0.4V
0.2V
0.0V
-0.2V
0.0s
0.1s
0.2s
0.3s
0.4s
0.5s
0.6s
0.7s
0.8s
0.9s
1.0s
x = 0.450s y = 0.005